In this seminar I briefly present some properties of binary star systems. I start with introduction about popularity in observing binary systems. I continue with mass transfer and concept of Roche Lobe. From mass transfer this seminar goes to formation of accretion disks. At the end of my seminar I explain transformation of gravitational potential energy to radiation and thermal energy with virial theorem.
INTRODUCTION

Universe is greatest and most mysterious puzzle for humankind from the beginning of our times. Evolution in technology and science made a big difference in observing this huge “empty space” and nowadays we can measure and explain many of It’s phenomena. Binary star systems are nowadays very popular to observe mainly for two reasons. In most cases binaries give more information about themselves than other astronomical objects. Secondly, most of stars are probably part of binary or multiple star systems and do transfer their masses at some point in their lifetime.

CONTENTS

ABSTRACT .................................................................................................................. 1
INTRODUCTION ......................................................................................................... 2
ROCHE LOBE OVERFLOW ...................................................................................... 3
  Binary systems ........................................................................................................ 3
  Roche lobe ........................................................................................................... 3
  Roche potential ..................................................................................................... 4
  Roche lobe quantitatively ..................................................................................... 5
DISK FORMATION ..................................................................................................... 6
VIRIAL THEOREM .................................................................................................... 8
CONCLUSION ........................................................................................................... 9
REFERENCES .......................................................................................................... 9

FIGURES

Figure 1: Binary system of normal star with mass M1 and compact star with mass M2. CM represents center of mass and a binary separation. Reproduced from [1]. ......................................................... 3
Figure 2: Roche potential of binary system with mass ration 0.25 represented with surface. Reproduced from [1]. ................................................................. 4
Figure 3: Two dimensional representation of Roche potential of two stars with mass ration 0.25 with all five Lagrangian points. Bold line represents edge of the Roche lobe. Reproduced from [1]. .......... 5
Figure 4: Roche lobes of stars in binary system, with inner Lagrangian point $L1$ and line segment $b1$ between $L1$ and primary star centre. Reproduced from [1]. ......................................................... 6
Figure 5: gas stream converging to compact star with spiral orbit. Reproduced from [1]. ............... 7
Figure 6: Artistic photo of quasar named QUASAR 311732. Author: Mark Kennedy. Reproduced from [6]. .......................... 8
ROCHE LOBE OVERFLOW

Binary systems
Two stars bonded together form so called binary system. Binary system can consist of different starts including white dwarfs and neutron stars. One of most known binary system is made of black hole named Cygnus X-1 and supergiant star. Characteristics of binary system depends on masses of both stars and binary separation between them.

Stars will orbit around each other with binary period, which can be calculated with third Kepler’s law:

\[ 4\pi a^3 = GM P^2. \]

Where \( M \) is sum of both star masses, \( G \) is the gravitational constant, \( a \) is the binary separation and \( P \) is binary period. Stars in binary systems can transfer mass in two different ways. One of the stars can eject some of it’s matter as a stellar wind which can be captured by companion star. Second mechanism is called Roche lobe overflow. It is basically a gravitational pull of matter from one star onto another. Falling matter accelerates quickly and develops high velocities. Because conservation of angular momentum captured mass does not fall straight into star but starts orbiting around it and forms so called accretion disc. Lose of gravitational potential energy transforms into kinetic energy and radiation. According to virial theorem half of the energy goes for kinetic energy and other half for radiation [1], [4].

Roche lobe
As said before there are two different ways of mass transfer in binary systems. Two stars can exchange mass via stellar wind or Roche lobe overflow. In this seminar we will focus on second one. Roche lobe also known as Roche limit was first discovered and studied by French astronomer and mathematician Edouard Roche. Simply put, Roche lobe is an area in which mass is gravitationally bonded to binary star system. To study or define Roche lobe we set a test particle in binary star system and observe it’s orbit. There are few assumption made in this model. First of all is that mass of a test particle is small compared to masses of both stars in systems and consequently does not perturb their orbits. Furthermore Roche model assumes that stars have circular orbits around each other. Last assumption in this model is that both stars are centrally condensed. In most cases this assumptions are good enough to match real situations with theory [1], [3].
Roche potential

Matter which is transferred between stars is in gas state. Conservation of momentum for gases is described with Euler equation:

$$\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v = -\nabla p + f.$$  \hspace{1cm} (1)

We can rewrite Euler equation () including all assumptions for Roche problem. Setting the reference frame to move with whole binary system gives us extra part with Coriolis force and centrifugal force. Part $2 \omega \wedge v$ represent Coriolis force per unit mass.

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla \phi_R - 2 \omega \wedge v - \frac{1}{\rho} \nabla p.$$ \hspace{1cm} (2)

Potential $-\nabla \phi_R$ is Roche potential and includes both gravitational and centrifugal component. $M_1$ and $M_2$ are masses of stars and $\omega$ is angular velocity of the system. Roche potential is given by:

$$\phi_R(r) = -\frac{GM_1}{|r-r_1|} - \frac{GM_2}{|r-r_2|} - \frac{1}{2} (\omega \wedge r)^2.$$ 

Figure 2: Roche potential of binary system with mass ratio 0.25 represented with surface. Reproduced from [1].

As we can see Roche potential has two valleys which represent both stars. Equipotential lines are directed by binary separation and mass ratio of the stars. Roche lobe is an area where mass is still gravitationally bonded to a star. The saddle point between two valleys is called inner Lagrange point and connects Roche lobes of binary stars. This point represents an energy minimum which gas needs to reach to start with mass transfer. There are five Lagrangian points where test particle can be in labile state. Roche lobe of binary system forms a dumbbell shape. Usually we name stars in binary system as primary and secondary star. Primary star is the compact one that steals mass from secondary star [1].
During evolutionary stage star can expand or shrink. When star expands so much that outer layers of gas reach inner Lagrangian point, mass transfer can begin. Gas from secondary star passes over the saddle potential and starts accelerating towards primary star. Any initial angular momentum affect the gas to start orbiting around primary star and disk formation begins. Accretion disks are efficient tools for extracting gravitational potential energy and transforming it into radiation.

Roche lobe quantitatively
Roche lobes in binary systems has shape of a drop. For quantitative discussion Roche lobe is approximated with sphere. The sphere has the same volume as actual lobe and it’s radius is good quantity for characterizing binary systems. Radius is determined by binary separation $a$ and stars mass ratio $q$. From here numerical approach needs to be made, however some analytic equations can be used.

\[
\text{Mass ratio of stars in binary system}: \quad q = \frac{M_2}{M_1}
\]

\[
\frac{r_{L2}}{a} = \frac{0.49q^{2/3}}{0.6q^{2/3} + \ln(1+q^{2/3})} \text{ for } 0 < q < \infty.
\]  

This equation gives as approximation for sphere radius of secondary star Roche lobe $r_{L2}$ more than 1% accurate over whole range of $q$. To calculate sphere’s radius for primary star, inverse mass ratio is used. Another often used formula for $0 \leq q \leq 0.8$ is Paczynski’s approximation:

\[
\frac{r_{L2}}{a} = 0.46224 \left(\frac{q}{1+q}\right)^{1/3}.
\]
Third approximation used for calculating Roche lobes ratio is:

\[ \frac{r_{L2}}{r_{L1}} = \left( \frac{M_2}{M_1} \right)^{0.45}. \]  

(5)

It is good for smaller mass ratios between 0.03 and 1.

Another parameter used for quantitative comparison of Roche lobes is distance between inner Lagrange’s point and primary star centre of mass [1], [5].

**DISK FORMATION**

Mass transfer via Roche lobe overflow directs that gas to go through inner lagrangian point. From primary star reference frame looks like gas is being squirted from rotating nozzle from secondary star. Rotating of this nozzle is consequence of binary period. We can evaluate components of gas stream velocities in non-rotating frame with next expressions:

\[ v_\perp \sim b_1 \omega. \]  

(6)

\[ b_1 \] is distance between primary mass centre and inner lagrangian point. \( \omega \) is angular velocity of binary and can be calculated as:

\[ \omega = \frac{2\pi}{P}. \]  

(7)

First equation gives perpendicular velocity to line between centres of stars. We can also approximate parallel velocity of gas stream. Gas is pushed through inner lagrangian point by pressure forces. That means that gas will have parallel velocity around speed of sound or less:

\[ v_{\parallel} \sim c_s. \]  

(8)

For star envelopes with temperatures below \( 10^5 \) K speed of sound cannot be much greater than 10 \( km \ s^{-1} \). Unless binary period is really long perpendicular velocity will be much greater than parallel velocity. Gas is squirted around primary star almost orthogonal to the line of star centres.

**Figure 4:** Roche lobes of stars in binary system, with inner Lagrangian point \( L_1 \) and line segment \( b_1 \) between \( L_1 \) and primary star centre. Reproduced from [1].
Gas orbiting inside Roche lobe of primary star is approximately circular. It’s initial orbit has to correspond to conservation of angular momentum. Using third Kepler’s law and classical mechanics of circulation we get:

$$rv = b^2 \omega.$$  \hspace{1cm} (9)

$$v = \left(\frac{GM_1}{r^2}\right)^{\frac{1}{2}}.$$  \hspace{1cm} (10)

$$r \left(\frac{GM_1}{r}\right)^{\frac{3}{2}} = b^2 \frac{2\pi}{p} = 2\pi b^2 \left(\frac{GM_1}{4\pi\rho}\right)^{\frac{1}{2}}.$$  \hspace{1cm} (11)

Initial radius or circularization radius of gas orbit depends only on binary separation and mass ration of stars. Circularization radius is usually smaller than Roche lobe’s radius, typically for factor 2-3 unless mass ratio is small. In some cases gas can be also prevented to orbit around primary star if primary’s radius is bigger that circularization radius. Transferred gas from secondary has to get rid of any angular momentum before reaches the primary surface. Gas interacts with itself, dissipating energy via shocks. Friction inside gas stream heats up gas particles and energy leaves system with radiation. In ideal accretion disk gas would converge to primary star in spiral orbits [1].

Figure 5: gas stream converging to compact star with spiral orbit. Reproduced from [1].

Understanding accretion disks goes way deeper and demands approach with viscous torques and magnetic hydrodynamics. However accretion disks are promising candidates for explaining quasars and active galactic nuclei which are one of the most powerful energy sources in universe [1].
VIRIAL THEOREM

Gravitational energy of star can be transformed into other energies. When transferred gas from secondary star falls deeper into gravitational field of primary star transforms its gravitational energy into heat and radiation. We can write gravitational energy of star with next equation:

$$E_g = \int_0^R \frac{GM(r)\rho(r)}{r} 4\pi r^2 dr.$$  \hspace{1cm} (12)

$$\frac{dp}{dr} = -\frac{GM(r)\rho(r)}{r^2}.$$  \hspace{1cm} (13)

$$E_g = \int_0^R r \frac{dp}{dr} 4\pi r^2 dr.$$  \hspace{1cm} (14)

First equation gives us gravitational energy of a star. Merging gravitational equation with hydrostatic equilibrium equations gives us third expression. After integrating with per partes method we get:

$$E_g = -12\pi \int_0^R -pr^2 dr.$$  

Using model of ideal gas gives us expression for thermal energy of a star and finally we can compare it with equation for gravitational energy. Thermal energy equal minus one half of gravitational energy.

$$w_n = n\frac{3}{2}kT.$$  \hspace{1cm} (15)

$$E_T = \int_0^R \frac{3}{2}p 4\pi r^2 dr = 6\pi \int_0^R pr^2 dr = -\frac{1}{2}E_g.$$  \hspace{1cm} (16)

Virial theorem explains that if star shrinks then transforms half of gravitational energy into thermal energy. Star eventually heats up with shrinking even if there would not be any nuclear reaction inside it. Other half of gravitational energy has to transform into radiation. With this concept with can understand accretion disk in another way [2].
CONCLUSION

This seminar summarizes some properties of the binary star systems. It explains some basic concepts of understanding mass transfer between two stars. These concepts can be also used in explaining binary systems where one of the stars can be replaced with neutron star or even black hole. Furthermore this seminar explains formation of accretion disks which are consequence of mass transfer. There are few approximation described in this paper that can be applied even in other braches of physics. Illustrating universe is very complex and often can not be done analytically. On the other hand approximations can describe it almost perfectly. It is clear that universe has always been biggest puzzle for scientist. Binary star systems are just one small piece that is interesting and exotic in every way.

REFERENCES