Abstract

In this seminar I introduce asteroseismology, which is a field of astrophysics studying stellar pulsations. One result of asteroseismology are the scaling relations, which allow us to calculate the mass and radius of pulsating stars from their power spectrum. I perform these calculations for one particular star and present my results.
1 Introduction

Understanding stellar structure and evolution is one of the main goals of modern astrophysics. But in order to test our stellar models, we need accurate experimental data. Until recently, photometry, astrometry, spectroscopy and interferometry were the only sources of our experimental data. These techniques, however, do not allow us to observe the interior of a star. A new technique, called helioseismology, has been developed, that enables us to do exactly that. Helioseismology is the study of Solar pulsations, and has so far greatly improved our understanding of the structure of the Sun (Aerts et al. 2010).

Helioseismology studies only the pulsations of the Sun. Asteroseismology, on the other hand, studies the pulsations of stars outside our Solar system. While observing pulsations of distant stars is more difficult than observing the Sun, asteroseismology has already presented us with some interesting results.

One such result are the scaling relations, which allow us to calculate the mass and radius of a star from its power spectrum. These scaling relations are the focus of this seminar. I first explain the basic principles of asteroseismology and show how they can be used to obtain the scaling relations, and then demonstrate their use by calculating the mass and radius of a red giant star HD 186355.

2 Stellar pulsations

As stars are three-dimensional objects, the oscillations in stars are described in three orthogonal directions, in this case the distance from the centre of the star $r$, longitude $\phi$, and co-latitude $\theta$. For spherically symmetrical stars, the oscillations in those directions are
described by the following equations (Aerts et al. 2010)\[1\]:

\[
\xi_r(r, \theta, \phi, t) = a(r)Y^m_l(\theta, \phi)\exp(-i2\pi\nu t),
\]

(1)

\[
\xi_\theta(r, \theta, \phi, t) = b(r)\frac{\partial Y^m_l(\theta, \phi)}{\partial \theta}\exp(-i2\pi\nu t),
\]

(2)

\[
\xi_\phi(r, \theta, \phi, t) = b(r)\frac{\sin \theta}{\sin \theta}\frac{\partial Y^m_l(\theta, \phi)}{\partial \phi}\exp(-i2\pi\nu t),
\]

(3)

where \(\xi\) is the displacement in the given direction, \(a\) and \(b\) are amplitudes, \(\nu\) is the oscillation frequency and \(Y^m_l\) are Laplace spherical harmonics, given by the following equation:

\[
Y^m_l(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P^m_l(\cos \theta) \exp(im\phi),
\]

(4)

where \(P^m_l\) are associated Legendre functions.

Figure 1: Surface pulsations with \(l = 3\). The white bands represent the nodes, red represents the part of the star that is moving in (heating), and blue represents the part of the star that is moving out (cooling). The columns show the star from different viewing angles (30, 60 and 90 degrees). The top row shows the mode with \(m = 0\), the second the mode with \(|m| = 1\), the third mode with \(|m| = 2\) and the bottom row the mode with \(|m| = 3\). Reproduced from Aerts et al. 2010 [1].
Stellar modes of pulsation can be described by three quantum numbers - $n$, which is also called the overtone of the mode and is related to the number of radial nodes, $l$, which is called the degree of the mode and tells us the number of present surface nodes, and $m$, also called the azimuthal order, whose absolute value $|m|$ tells us the number of surface nodes that are lines of longitude. From this last definition follows that $l - |m|$ gives us the number of surface nodes that are lines of co-latitude. Values of $m$ range from $-l$ to $l$, meaning that there are $2l + 1$ modes for each degree $l$.

Figure 1 illustrates an example of surface pulsations with $l = 3$.

These pulsations can, and have been detected experimentally. When a star is pulsating, its surface expands and contracts, as illustrated in Figure 1. The parts of the star that are contracting heat up, and become brighter. Conversely, the parts of the star that are expanding cool down and become dimmer. Since what we detect when observing a star is its total brightness, it is intuitively clear that the effects of pulsations with a high degree $l$ will cancel out. In the radial modes of pulsation ($l = 0$), however, this cancelling effect is not present. Stellar pulsations can therefore be detected as oscillations in the stellar light curve. Figure 2 represents a light curve of the star HD 186355, on which the effect of pulsations is clearly visible. In section 4, I will use this light curve to calculate the mass and radius of HD 186355.

![Figure 2: Light curve of HD 186355, and a zoomed in light curve of the same star. Oscillations are clearly visible in both light curves.](image)

### 2.1 $p$ modes and $g$ modes

Depending on which force is the restoring force of the pulsation, we distinguish two modes of stellar pulsations - the pressure modes, or $p$ modes, and gravity modes, or $g$ modes. In $p$ modes, the restoring force is pressure. In $g$ modes, the restoring force is buoyancy.
As a consequence, the \( g \) modes cannot propagate through convective layers of stars, and are, in case of Solar like pulsators, trapped beneath the convective envelope of the star and cannot be observed at the surface.

The pressure modes, on the other hand, are acoustic waves, and therefore can propagate through convective regions of a star. When a \( p \) mode wave propagates through the star, the part of the wave that is closer to the center of the star will be in a higher temperature environment than the part of the wave closer to the stellar surface. Acoustic wave propagates faster in high temperature environment than in low temperature environment (cf. Section 3), which causes the waves with non-radial components (waves with \( l \neq 0 \)) to refract towards the surface of the star. Figure 3 illustrates the behaviour of \( p \) and \( g \) waves inside a Solar-like star.

![Figure 3: Propagation of \( p \) mode (a) and \( g \) mode waves inside a Solar-like star. Adapted from Aerts et al. 2010[1].](image)

By observing the frequencies of pulsation of \( p \) modes with different values of \( l \), we are therefore indirectly observing the sound speed, and consequently the temperature profile, inside a star.

The frequencies of \( p \) mode waves are given by the following equation (Tassoul 1980)[2]

\[
\nu_{n,l} = \Delta \nu (n + l/2 + \alpha) + \epsilon_{n,l},
\]

where \( \alpha \) is a constant, \( \epsilon \) is a small correction and \( \Delta \nu \) is called large separation, and can be estimated as:

\[
\Delta \nu = \left( 2 \int_0^R \frac{d r}{c(r)} \right)^{-1}.
\]
On the power spectrum of a star, the large separation is therefore the separation between two modes with the same value of \( l \) and the value of \( n \) increased or decreased by one:

\[
\nu_{n+1,l} - \nu_{n,l} = \Delta \nu. \tag{7}
\]

Equation (5) follows from the asymptotic theory of stellar oscillations. Equations of stellar oscillations are in general very difficult to solve. The asymptotic theory simplifies them by considering only oscillations with high values of either \( n \) or \( l \), which is a valid approximation in the case of \( p \) mode waves. Detailed derivation can be found in Tassoul (1980)\cite{2} or Aerts et al. (2010).\cite{1}

3 The scaling relations

Asteroseismology allows us to calculate the mass and radius of a star with the help of scaling relations, which relate the observed frequencies in the power spectrum of a star to the mass and radius of the star. To understand how, let us first take another look at Equation (6). The adiabatic sound speed is given by:

\[
c = \sqrt{\frac{\gamma p}{\rho}}, \tag{8}
\]

which, under the approximation that the star can be described as an ideal gas, simplifies to:

\[
c = \sqrt{\frac{kT}{\mu}}, \tag{9}
\]

where \( k \) and \( \gamma \) are constants and \( \mu \) is the average mass per particle. While \( \mu \) is technically not constant throughout the star, it can be approximated as a constant in the case of \( p \) mode waves. The sound speed is therefore a function of temperature only, and \( \Delta \nu \) scales approximately as:

\[
\Delta \nu \propto \sqrt{\frac{\langle T \rangle}{R}}, \tag{10}
\]

where \( \langle T \rangle \) is the average temperature. The average temperature of a star can be estimated as \( \langle T \rangle \propto \frac{M}{R} \) (Kippenhahn and Wigert 1990)\cite{3}. Substituting this relation into Equation (10) gives us the first scaling relation:

\[
\Delta \nu \propto M^{\frac{1}{2}} R^{-\frac{3}{2}}. \tag{11}
\]

We can see from this scaling relation that \( \Delta \nu \) is proportional to the square root of the average stellar density, which is given as \( \langle \rho \rangle = \frac{3M}{4\pi R^3} \).
The second scaling relation can be derived from the photospheric acoustic cut-off frequency \( \nu_c \). A cut-off frequency is the frequency above which the waves no longer propagate. The acoustic cut-off frequency is given by the following equation:

\[
\nu_c = \frac{c}{2H}, \tag{12}
\]

where \( c \) is the speed of sound and \( H \) is the pressure scale height, defined as the distance over which the value of pressure falls by a factor of \( e \). The pressure scale height can be calculated from its definition - if we assume that the atmosphere of the star is described by the ideal gas law and apply the barometric law, we get the following equation:

\[
\frac{P(H)}{P(0)} = e^{-1} = e^{-mgH/(kT)}. \tag{13}
\]

It follows directly that the pressure scale height is:

\[
H = \frac{kT}{mg}, \tag{14}
\]

where \( T \) is the temperature, \( g \) is gravitational acceleration, and \( k \) and \( m \) are constants. Substituting \( H \) from Equation (14) into Equation (12), and taking into account that the sound speed is given by Equation (9), gives us the following relation:

\[
\nu_c \propto \frac{g}{\sqrt{T}}. \tag{15}
\]

Experimental results show that the acoustic cut-off frequency scales with the frequency of maximum power \( \nu_{\text{max}} \), which we can observe as the peak of the envelope of the power spectrum of a star, giving us the second scaling relation:

\[
\nu_{\text{max}} \propto \frac{M}{R^2 \sqrt{T_{\text{eff}}}}. \tag{16}
\]

As these scaling relations hold for Solar-like pulsators in general, they must also hold for our Sun. Combining them and dividing them by the mass and radius of the Sun yields the following two expressions:

\[
\frac{M}{M_{\odot}} \approx \left( \frac{\nu_{\text{max}}}{\nu_{\text{max,\odot}}} \right)^3 \left( \frac{\Delta \nu}{\Delta \nu_{\odot}} \right)^{-4} \left( \frac{T_{\text{eff}}}{T_{\text{eff,\odot}}} \right)^{\frac{3}{2}}, \tag{17}
\]

\[
\frac{R}{R_{\odot}} \approx \left( \frac{\nu_{\text{max}}}{\nu_{\text{max,\odot}}} \right) \left( \frac{\Delta \nu}{\Delta \nu_{\odot}} \right)^{-2} \left( \frac{T_{\text{eff}}}{T_{\text{eff,\odot}}} \right)^{\frac{1}{2}}. \tag{18}
\]
The symbol $\odot$ indicates the value of the quantity for the Sun. As the mass, radius, temperature, $\nu_{\text{max}}$ and $\Delta \nu$ of the Sun are known, all that is needed to calculate the mass and radius of a pulsating star is to determine the parameters $\nu_{\text{max}}$ and $\Delta \nu$ from its power spectrum. Figure 4 illustrates how the mass and radius of a star change with frequencies $\nu_{\text{max}}$ and $\Delta \nu$. 

![Figure 4](image)

Figure 4: The figure shows how mass and radius of a star change with frequencies $\nu_{\text{max}}$ and $\Delta \nu$. Numbers on the plot represent the mass and radius of a star in units of Solar mass and radius.

4 Calculating the mass and radius of HD 186355

4.1 Power spectrum

Equations (17) and (18) enable us to calculate the mass and radius of a star. In this sections I will demonstrate how - I will calculate the power spectrum of a red giant HD 186355 from its light curve, determine the parameters $\Delta \nu$ and $\nu_{\text{max}}$, and calculate its mass and radius. For my analysis I have used the observations acquired by the *Kepler* telescope, which are publicly available through the MAST (Mikulski Archive for Space Telescopes) archive[5].

The first step is to perform a discrete Fourier transform of the stellar light curve to get the power spectrum. Results are presented in Figure 5a.

To determine the values of parameters $\nu_{\text{max}}$ and $\Delta \nu$, the peak of the power spectrum has to be analysed. The peak we are interested in lies around the frequency $\nu = 100\mu Hz$. A power spectrum zoomed in around this frequency is shown in Figure 5b.

The frequency of maximum power $\nu_{\text{max}}$ and large separation $\Delta \nu$ now need to be determined from the power spectrum. Note that even though $\nu_{\text{max}}$ is called the frequency of maximum power, we are actually not interested in the frequency of the highest peak. Instead, a Lorentz
distribution is fitted to the group of frequencies around this peak. $\nu_{\text{max}}$ is the frequency of the maximum of this Lorentz distribution. To find $\Delta \nu$, recall Equation (7). The large separation $\Delta \nu$ is the separation between any two modes with the same value of $l$, and a value of $n$ that is either increased or decreased by one. That means that peaks in the spectrum should appear at regular frequency separations. Finding such a frequency separation means that we have found $\Delta \nu$.

From the power spectrum presented in Figure 5, I have determined $\nu_{\text{max}}$ and $\Delta \nu$ to be $(106 \pm 2) \mu Hz$ and $(10 \pm 1) \mu Hz$ respectively.

### 4.2 Calculation

Equations (17) and (18) can now be used. $\nu_{\text{max}}$ and $\Delta \nu$ have been determined from the power spectrum, and all other parameters in these equations are known. The value of $\nu_{\text{max,\odot}}$ is $3090 \mu Hz$, the value of $\Delta \nu_{\odot}$ is $135.1 \mu Hz$ (Chaplin et al. 2014)[6], the temperature of the Sun is $5778 K$, and the temperature of HD 186355 is given by *Kepler* as $4867 K$[5]. For simplicity, I will present the mass and radius of HD 186355 in units of solar mass and radius:

$$ M = (1.1 \pm 0.5) M_\odot \quad R = (5.7 \pm 1.2) R_\odot $$

(19)

Jiang et al. have calculated the mass of HD 186355 to be $(1.41 \pm 0.14) M_\odot$, using a more advanced version of the method I have used in this seminar (Jiang et al. 2011)[7]. Performing the zeta score calculation gives the value of $\zeta = 1.15$, which means that the results are in agreement. The relatively high uncertainties of my calculations are the result of the large uncertainty of $\Delta \nu$, which is difficult to determine accurately without the use of advanced stellar models.
5 Conclusion

The scaling relations of asteroseismology are a relatively simple, yet very effective method for calculating masses and radii of pulsating stars. With the help of scaling relations, I have calculated the mass and radius of a red giant HD 186355. The results seem sensible for a red giant star, as the stars of this type do have significantly lower densities than the main sequence stars. The mass I have calculated is in agreement with the calculations by Jiang et al.[7] who have calculated the mass of HD 186355 using a more advanced version of the method I have used in my seminar.

References


