Abstract

In this seminar, magnetohydrodynamic (MHD) equilibrium and stability of the plasma are presented. First, we define plasma and introduce some of the basic properties of it, as Debye shielding and plasma oscillations. Further on we introduce kinetic approach of describing plasma and then its simplified, but still accurate enough macroscopic derivation named two-fluid model. Plasma can also be treated as a single conducting fluid, which is carried out in magnetohydrodynamic model. It is the least accurate macroscopic model, but still it can correctly describe equilibrium and stability in real fusion plasmas. In one of the most promising designs, tokamak, plasma must be in radial pressure balance and toroidal force balance to remain in equilibrium, which is achieved by different pinch configurations. What is more, various pinches can also excite instabilities at different growth levels, which also must be properly investigated.
1 Introduction

Plasma research gained on its importance over the past decades due to idea of fusion being a new source of energy produced in a fusion reactor. In these machines hydrogen plasma is confined and heated to temperatures high enough to achieve and sustain the fusion reaction. The most promising and the easiest to achieve is reaction between deuterium and tritium:

\[
\frac{2}{1}D + \frac{3}{1}T \rightarrow \frac{4}{2}He + \frac{1}{0}n.
\]

Released energy is distributed between alpha particle (14.1 MeV) and neutron (3.5 MeV), where only charged alpha particles are confined by magnetic field and heat the plasma inside the reactor [3].

There are several ways to describe the plasma behaviour, but the most useful and the least time consuming for calculations is magnetohydrodynamic (MHD) model, which will be described in detail in this seminar. Major issues as macroscopic equilibrium and stability of the plasma can be investigated with this model.

2 Basic description of plasma

Plasma is one of the four fundamental states of matter in the Universe in addition to solids, liquids and gases [1]. It is an ionized gas, originating from the neutral gas raised to a sufficiently high temperature, subjected to electric fields or photoionization process (i.e. absorption of incident photons with energy equal or greater then the ionization energy of absorbing atom). In the absence of external disturbances a plasma is macroscopically neutral, which means the net electric charge is zero. When a gas is ionized, its behaviour is dominated by the electromagnetic forces acting on the free ions and electrons, and it begins to conduct electricity which implies that currents can flow in plasma. The charged particles inside plasma interact with electromagnetic fields and electrical currents can in turn produce electromagnetic fields. The ability of an ionized gas to sustain electric current is important in the presence of magnetic...
field, which yields a Lorentz force on charged particles that can dominate the gas dynamics and confine the plasma [2]. Some of the basic plasma parameters are: the particle density $n$, the temperature $T$ of each species, the collision frequency $\nu$ of each species and the plasma frequency $\omega_p$.

2.1 Debye shielding

One of the fundamental properties of a plasma is its tendency to maintain electric charge neutrality. Departures from neutrality can occur only over distances in which a balance is obtained between the thermal particle energy and the electrostatic potential energy resulting from any charge separation, which tends to restore the electrical neutrality. This characteristic length is called the Debye length. The charged particles arrange themselves in a way to effectively shield any electrostatic fields caused by charge imbalance within this range [2]. If we slowly insert a plate electrode with potential $\phi_0$ inside the plasma and calculate the potential $\phi$ from Poisson’s equation in one dimension, we get

$$\phi(x) = \phi_0 e^{-\frac{x}{\lambda_D}},$$

where $\lambda_D$ is the Debye length. It is defined with the summation over all species $\alpha$ participating in Debye shielding as

$$\frac{1}{\lambda_D^2} = \sum_{\alpha} \frac{1}{\lambda_{\alpha}^2},$$

where $\lambda_{\alpha}$ is given by

$$\lambda_{\alpha} = \sqrt{\frac{\epsilon_0 k_B T_i}{n_0 e_i^2}}.$$  

In (3), $T_\alpha$, $e_\alpha$ and $n_0$ represent temperature, charge and equilibrium number density in undisturbed plasma. $\epsilon_0$ and $k_B$ are the dielectric constant and Boltzmann constant. A necessary requirement for the existence of plasma is $L \gg \lambda_D$, if $L$ is a characteristic dimension of the plasma. Otherwise there is not sufficient space for the collecting shielding effect to take place. It is also necessary that the number of electrons inside a Debye sphere is very large, which is defined by

$$n_e \lambda_D^3 \gg 1.$$  

Condition (4) is another criterion for the definition of the plasma.

2.2 Plasma oscillations

Electric field resulting from charge separation accelerates electrons in attempt to restore the charge neutrality and causes oscillations when electrons are moving around equilibrium position because of their inertia. These oscillations are high frequency oscillations. Ions, due to their heavy mass, are unable to follow the motions of electrons so we can neglect their motion in perturbation. We also assume a very small electron density perturbation such that [2]

$$n_e(\vec{r}, t) = n_0 + n'_e(\vec{r}, t),$$

where $n_0$ is a number density in equilibrium and $|n'_e| \ll n_0$. From above considerations and further derivation [2] we obtain an equation with harmonic time variation of perturbation

$$\frac{\partial^2 n'_e}{\partial t^2} + \omega_p^2 n'_e = 0,$$

where $\omega_p$ is a natural frequency of oscillation known as a plasma frequency:
\[ \omega_p = \sqrt{\frac{n e^2}{m_e \epsilon_0}}. \]  

(7)

\( m_e \) is a mass of electrons. Collisions between electrons and neutral particles tend to damp oscillations and gradually diminish their amplitude. To keep playing an important role, it is necessary for oscillations that the electron-neutral collision frequency \( \omega_{en} \) is smaller than plasma frequency,

\[ \omega_p \gg \omega_{en}. \]  

(8)

Otherwise, electrons will be forced by collisions to be in equilibrium with neutrals and the medium is treated as a neutral gas. (8) is another criterion for the existence of plasma.

3 Kinetic theory

A plasma is a system that contains a large number of interacting charged particles and for its analysis it is the most appropriate to use a statistical description since we are more interested in macroscopic properties than in behaviour of individual particle. All physically interesting information about the system is contained in the distribution function \( f_\alpha(\vec{r}, \vec{v}, t) \), which is defined as [2]

\[ f_\alpha(\vec{r}, \vec{v}, t) = \frac{d^6 n_\alpha(\vec{r}, \vec{v}, t)}{d^3 r d^3 v}, \]  

(9)

where \( d^6 n_\alpha(\vec{r}, \vec{v}, t) \) denotes the number of particles type \( \alpha \) inside a small finite element of volume \( d^3 v \) around the coordinates \( \vec{r}, \vec{v} \) at the instant \( t \). Once we know the distribution function, we can calculate average value of some physical property \( \chi(\vec{r}, \vec{v}, t) \) of particles type \( \alpha \) as [2]

\[ \langle \chi(\vec{r}, \vec{v}, t) \rangle_\alpha = \frac{1}{n_\alpha(\vec{r}, t)} \int_v \chi(\vec{r}, \vec{v}, t)f_\alpha(\vec{r}, \vec{v}, t)d^3 v. \]  

(10)

In general, distribution function can be obtained by solving the Boltzmann equation, which is a differential equation that governs the temporal and spatial variation \( (\vec{r}, \vec{v} \text{ and } t) \) of the distribution function [2]:

\[ \frac{\partial f_\alpha}{\partial t} + \vec{r} \cdot \vec{\nabla} f_\alpha + \frac{\vec{F}}{m_\alpha} \cdot \nabla_v f_\alpha = \left( \frac{\partial f_\alpha}{\partial t} \right)_{\text{coll}}. \]  

(11)

\( \left( \frac{\partial f_\alpha}{\partial t} \right)_{\text{coll}} \) represents the rate of change of \( f_\alpha(\vec{r}, \vec{v}, t) \) due to collisions and \( \vec{F} \) is defined as

\[ \vec{F} = \vec{F}_{\text{ext}} + q_\alpha(\vec{E}_i + \vec{v} \times \vec{B}_i). \]  

(12)

\( \vec{F}_{\text{ext}} \) includes the Lorentz force associated with external electric and magnetic fields. \( \vec{E}_i \) and \( \vec{B}_i \) are internal fields due to the presence and motion of all charged particles inside the plasma.

3.1 The equilibrium state

The equilibrium distribution function is a time independent solution of Boltzmann equation without any external forces and spatial gradients in the particle number density. It is known as the Maxwell-Boltzmann distribution function and can be written as [2]

\[ f_\alpha(\vec{v}) = n_\alpha \left( \frac{m_\alpha}{2\pi k_B T} \right)^{\frac{3}{2}} \exp \left( - \frac{m_\alpha (\vec{v} - \vec{u}_\alpha)^2}{2k_B T} \right) \]  

(13)

\( n_\alpha, m_\alpha, T_\alpha \) and \( \vec{u} \) are the particle’s density, mass, temperature and mean velocity. In many situations a gas of interest is not in equilibrium, but it is not far from it. It is a good approximation then if
we consider that around every point in the gas there is an equilibrium described by a local Maxwell-Boltzmann distribution function in which \( n, T \) and \( \bar{u} \) are slowly varying with \( r \) and \( t \).

4 The fluid model

Kinetic approach by solving the Boltzmann equation is the most accurate way to calculate the distribution function, but also a matter of great difficulty. Another approach are macroscopic transport equations for each plasma species governing the temporal and spatial variation of macroscopic variables, which can be obtained in a simpler way, that is by taking moments of the Boltzmann equation. The first three moments of the Boltzmann equation for each species \( \alpha \) are gained by multiplying it by \( m_\alpha, m_\alpha \bar{u} \) and \( \frac{m_\alpha v^2}{2} \), respectively, and integrating over all of velocity space. They give us the equation of conservation of mass (14), the momentum (15) and the energy for each of the species of particles and measurable physical quantities can be obtained from them. On the other hand, due to greater simplicity, at each stage of these moments, resulting set of equations is never complete in a way that the number of unknowns is always greater than the number of equations, which is not sufficient to determine all macroscopic variables in them. Thus, we have to introduce some simplifying assumptions concerning the highest moment of the distribution function in our set of equations. For instance, one of the simplest closed systems known as the cold plasma model, where temperature of plasma is set to zero, contains only equations of conservation of mass and momentum for each species [2]:

\[
\frac{\partial \rho_\alpha}{\partial t} + \bar{\nabla} \cdot (\rho_\alpha \bar{u}_\alpha) = 0,
\]

(14)

\[
\rho_\alpha \left( \frac{\bar{u}_\alpha}{\partial t} + (\bar{u}_\alpha \cdot \bar{\nabla}) \bar{u}_\alpha \right) + \bar{\nabla} \cdot \bar{p}_\alpha - n_\alpha q_\alpha (\bar{E} + \bar{u} \times \bar{B} - \bar{A}_\alpha) = 0,
\]

(15)

where \( \rho_\alpha, \bar{u}_\alpha \) and \( \bar{A}_\alpha \) are the mass density, mean velocity and the collision term for momentum transfer between species. The highest momentum appearing in the momentum equation, tensor \( \bar{p} \), is set to zero in this model to close the set of equations. That means that the effects due to the thermal motion of particles and the force due to the divergence of the kinetic pressure tensor are neglected. Here we also assumed the absence of processes leading to production and loss of particles.

5 Magnetohydrodynamic model

Plasma can also be treated as a single conducting fluid, without specifying its various individual species. In the most general form, known as a magnetohydrodynamic (MHD) model, each macroscopic variable is combined with contributions of various particles in plasma as their weighted mean value. In practice, several simplifying approximations are considered, based on physical arguments, which permit the elimination of some terms in the equations. Since MHD equations are easier to solve, they can be used to study real geometries.

First, we are introducing single-fluid variables into our macroscopic transport equations for multiple species. We assume that our plasma consists mainly from ions and electrons. Since mass of electrons is much smaller than the mass of ions, total mass density \( \rho \) corresponds mainly to that of the ions [3]:

\[
\rho = m_i n_i + m_e n_e,
\]

(16)

where \( n_e = n_i = n \) because of charge-neutrality. This consideration is due to the characteristic length much larger than the Debye length \( (L \gg \lambda_D) \) and frequency much smaller than the plasma characteristic frequency \( (\omega \sim v_T i / L \ll \omega_{pe} \sim \omega_p) \). Next, the electron mean velocity can be defined in terms of the current density \( \bar{J} = n_e (\bar{u}_i - \bar{u}_e) \) as [3]

\[
\bar{u}_e \approx \bar{u}_i - \frac{\bar{J}}{qn},
\]

(17)
where $\vec{u} \approx \vec{u}_i$, because of $m_e \ll m_i$. Finally the pressure in MHD model is the total electron and ion pressure: $p = p_e + p_i$.

### 5.1 Conservation equations

Conservation of mass equations for different species can be easily obtained in single-fluid model as [3]

$$\frac{dp}{dt} + p \vec{\nabla} \cdot \vec{u} = 0,$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}$ is the total time derivative moving with the ion fluid. The most important MHD equation due to its description of basic force balance of the plasma is the equation of motion [3]:

$$\rho \frac{d\vec{u}}{dt} = \vec{J} \times \vec{B} - \vec{\nabla} p.$$  \hspace{1cm} (19)

Electric fields from (15) have cancelled because of assumption of charge neutrality and collisional terms due to the conservation of total momentum of particles in plasma. The left side of (19) represents the inertial force, $\vec{J} \times \vec{B}$ magnetic field force to confine plasma and $\vec{\nabla} p$ pressure gradient force that causes the expansion of plasma. To obtain equilibrium in steady state, magnetic force needs to balance the pressure gradient force, which means that $J \sim \frac{p}{LB} \ll 1$ [3].

From the transformation of the electron momentum equation into a single-fluid Ohm’s law, a relation between electric field and current can be obtained. Electron inertia term can be neglected since electron response time is essentially faster compared to the characteristic MHD frequency ($\omega \ll \omega_{ce} \sim \omega_p$), because of $m_e \ll m_i$. Introducing single fluid variables in electron momentum equation yields [3]:

$$\vec{E} + \vec{u} \times \vec{B} = \frac{1}{q_i} (\vec{J} \times \vec{B} - \vec{\nabla} p) + \eta \vec{J},$$  \hspace{1cm} (20)

which is known as a generalized Ohm’s law, where $\eta$ represents the resistivity of plasma. Terms on the right can be neglected in the ideal MHD regime, which means that plasma behaves as an ideal conducting material.

In the MHD regime the energy equations simplify considerably. In the model we assume the adiabatic system, so there is no heat exchange across the system. Taking into account this assumption, we get [3]

$$\frac{d}{dt} \left( \frac{p}{p_i} \right) = 0,$$

where $\gamma$ is a ratio between specific heat at constant pressure and constant volume. With $N$ being the degree of freedom, it is defined as $\gamma = \frac{2N}{N}$. For monoatomic gas $N = 3$ and thus $\gamma = \frac{5}{3}$.

### 5.2 Electrodynamic equations

Beside transport equations we need to introduce electrodynamic equations to obtain a complete set of equations. They relate $\vec{E}, \vec{B}, \vec{J}$ and charge density $\sigma$ through Maxwell’s equations, with some approximations. Normally we neglect the displacement current $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ in $\vec{\nabla} \times \vec{B}$ term due to non-relativistic plasma flows ($u_{T_i} \ll c$). We also assume the electric neutrality and therefore charge density is set equal to zero in term $\vec{\nabla} \cdot \vec{E}$.

A complete set of transport and electrodynamic equations defining the MHD model can be summarized as [3]
\[
\begin{align*}
\frac{dp}{dt} + \rho \vec{V} \cdot \vec{u} &= 0 \\
\vec{E} + \vec{u} \times \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\end{align*}
\]

\[
\begin{align*}
\frac{d\vec{u}}{dt} &= \vec{J} \times \vec{B} - \vec{\nabla} p \\
\frac{d}{dt} (p \gamma) &= 0 \\
\vec{\nabla} \times \vec{E} &= -\mu_0 \frac{d\vec{B}}{dt} \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \cdot \vec{E} &= 0
\end{align*}
\]

This model will be used for further analysis of macroscopic equilibrium and stability of hot fusion plasma.

6 MHD equilibrium

One of the most promising designs of a fusion power plant nowadays are tokamaks, which are based on a torus shaped vacuum chamber, where plasma inside is confined with strong magnetic fields. These fields isolate hot, over 100 million degrees heated, plasma from the first wall of the chamber and thus keep the walls cool and prevent damages on its surface. To hold the plasma on desired location inside the chamber, plasma must be in equilibrium. MHD equilibrium is defined by a time independent, static \((\vec{u} = 0)\) plasma, with all identified forces exactly in balance. Ideal MHD equation of motion gives us relation \[3\]
\[
\vec{J} \times \vec{B} = \vec{\nabla} p.
\]

If we form the dot product of (22) by \(\vec{B}\) or \(\vec{J}\) we get \(\vec{B} \cdot \vec{\nabla} p = 0\) and \(\vec{J} \cdot \vec{\nabla} p = 0\). This implicates that magnetic and current lines lie on the surfaces of constant pressure with the arbitrary angle between \(\vec{B}\) and \(\vec{J}\), forming a set of closed nested toroidal surfaces. Magnetic forces that hold plasma together are combination of magnetic pressure and tension and they have to balance the plasma pressure to create an equilibrium force balance, which is a combination of radial and toroidal force balance.

6.1 Radial pressure balance

Let’s use a model of toroidal configuration, which we cut at an arbitrary poloidal plane, straighten out the torus and transform it into equivalent straight cylinder. Thus all quantities depend only upon the minor radius \(r\). With this model we can investigate several different configurations that provide radial pressure balance: \(\theta\)-pinch, \(Z\)-pinch and screw pinch.

In \(\theta\)-pinch configuration current flows in the coil as shown in figure 1. This produces a magnetic field \(B_0\) inside the coil and a poloidal current \(J_\theta\) is induced in a direction to cancel the applied field. From ideal MHD electrodynamic equations \((\vec{\nabla} \cdot \vec{B} = 0, \mu_0 J_\theta = -\frac{d\vec{B}}{dt})\) and integrating the equation of motion (19), the basic radial pressure balance can be obtained [3]:

\[
\begin{align*}
\frac{d}{dt} \left( p(r) + \frac{B_\theta^2(r)}{2\mu_0} \right) &= \frac{B_0^2}{2\mu_0} p(r) \\
\end{align*}
\]

(23) means that at any radial position \(r\), sum of local plasma pressure and internal magnetic pressure is equal to the applied magnetic pressure \(\frac{B_0^2}{2\mu_0}\), \(p\) is assumed to be zero at the plasma boundary. Quantity that measures how effective the applied fields \((B_0)\) are at confining plasma pressure is called beta ratio \(\beta\) and must be between limits 0 and 1 [3]:

\[
\beta = \frac{p}{\frac{B_0^2}{2\mu_0}} < 1.
\]

Another configuration that is capable of providing the radial pressure balance is the so called \(Z\)-pinch, where the current in \(z\) direction produces a poloidal magnetic field \(B_\theta\) as shown in figure 2. In this configuration, proceeding the same as for \(\theta\) pinch, we are integrating the relation [3]
Figure 1: Schematic diagram of a $\theta$-pinch \[3\].

Figure 2: Schematic diagram of a $Z$-pinch \[3\].

Figure 3: The radial forces on the $Z$-pinch \[3\].

$$
\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{2\mu_0 r} = 0,
$$

where the terms from left to right correspond to the plasma pressure, magnetic pressure and magnetic tension. If we plot all contributing forces as a function of radius (figure 3), we see that near the outer edge of plasma only the magnetic tension force is confining and provides balance. Beta ratio does not have any flexibility in $Z$-pinch, for all profiles it is equal to $\beta = 1$.

Configuration that is a combination of $\theta$-pinch and $Z$-pinch is called a screw pinch. Here the magnetic lines are twisted around the surface giving the appearance of a screw thread. General radial pressure balance, where $\vec{B} = B_\theta \hat{e}_\theta + B_z \hat{e}_z$ and $\vec{J} = J_\theta \hat{e}_\theta + J_z \hat{e}_z$, is given by \[3\]

$$
\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}ight) + \frac{B_\theta^2}{2\mu_0 r} = 0.
$$

We observe that in general one can specify two arbitrary functions (for example $B_\theta(r)$ and $B_z(r)$) and then determine the third function with MHD (in this case $p(r)$). Thus the screw pinch enables a wide range of configurations, for example independently programming coil currents in toroidal and poloidal field circuits, and is therefore promising configuration for the tokamak concept.

### 6.2 Toroidal force balance

A fusion plasma must be in a shape of torus in order to avoid end losses. Thus we bend previously defined straight cylinder into a torus, which generates unavoidable forces by toroidal and poloidal magnetic fields directed outwardly along the direction of the major radius $R$. To hold plasma in a toroidal force balance and prevent it from striking the first wall, we must apply some additional forces to resist the outward forces. In the analysis below all forces are presented \[3\].
The hoop force is generated by the current flowing in the circular wire, which corresponds to the toroidal current in plasma inside Z-pinch bended into a torus. Now let’s divide plasma into two halves as seen in figure 4. Surface on the inner side \((S_1)\) is clearly smaller than the one on the outer side \((S_2)\), so \(S_1 < S_2\). Magnetic field is greater on the inner side \(B_1 > B_2\) since the given amount of poloidal flux \(\psi\) on the outer side must be squeezed into smaller cross sectional area on the inside. Focusing on the magnetic tension force on each half, \(F_1 = \frac{B_1^2}{2\mu_0} S_1\) and \(F_2 = \frac{B_2^2}{2\mu_0} S_2\), and they are pointed towards the plasma. \(F_1 > F_2\) due to the domination of quadratic dependence of \(B\), which create net outward force along \(R\).

The tire tube force originates from the plasma pressure \(p\) and is thus generated in both, Z-pinch and \(\theta\)-pinch configuration. On a surface of constant pressure in tokamak as shown in figure 5, the expansion force on inner half is equal to \(F_1 = pS_1\) and on the outer half \(F_2 = pS_2\). The net force points outward along \(R\) due to the difference of size of the surface area \((S_1 < S_2)\).

![Figure 4: Qualitative picture of the hoop force][3].

![Figure 5: Qualitative picture of the tube force][3].

Since these two forces above push plasma outwards in a direction along \(R\), an inward pointed force is required to establish toroidal force balance. Restoring force can be produced by using a perfectly conducting wall or by applying an external vertical field. Perfectly conducting wall method is demonstrated in figure 7. When plasma moves outwards along \(R\), eddy currents\(^1\) are induced in the wall and magnetic field lines between the plasma and the outer wall become compressed, which means that the magnitude of poloidal field is increased. At some point the magnetic tension force on the outer side increases so much that it can balance the outward forces. This method doesn’t work in a \(\theta\)-pinch configuration since there is no poloidal field to trap. In practice, even for other pinches balance by this method is almost impossible to achieve. The reason is in high temperature of plasma and in release of large amounts of neutrons through the walls, which don’t allow the wall to be superconducting. With a finite conductivity, poloidal field lines can diffuse through the wall and plasma shifts outwards \([3]\).

More practical way to achieve toroidal balance is to add an externally applied vertical field as shown in figure 8. If we add a uniform vertical field to the poloidal field, which has a bigger magnitude on the inside of the torus due to \(1/R\) dependence, we can achieve the balance by adjusting the amplitude of vertical field. In this configuration first wall needs to have a finite conductivity to allow vertical field to penetrate through the plasma. This technique can not be used in pure \(\theta\) pinch as there the current and vertical magnetic field lie in a poloidal plane and thus no force is produced along \(R\) \([3]\).

### 7 MHD-macroscopic stability

Another application of the MHD model concerns the macroscopic stability of plasma. First we assume that MHD equilibrium has been found, which provides a good plasma confinement. Then we study whether the plasma, which is perturbed away from the equilibrium, would return to its initial position as time progresses. If it returns back or at worst, oscillates around its equilibrium position, we consider

\(^1\) Circular electric currents induced within conductors by a changing magnetic field in the conductor, due to Faraday’s law of induction \([4]\).
it as stable. If the initial perturbation continues to grow from the equilibrium position, it is considered as unstable. These instabilities can lead to huge losses of plasma and damages of the first wall and thus have to be avoided. That can be done by limiting the amount of pressure or toroidal current [3].

7.1 The frozen-in-field-line concept

The frozen-in-field-line concept is an important ideal MHD property, which shows that perpendicular to the magnetic field lines, plasma and field lines move together. Let’s consider the flux $\psi(t)$ defined as [3]

$$\psi(t) = \int \vec{B} \cdot \vec{n} dS,$$  \hspace{1cm} (27)

where we integrate over an open surface $S$ with normal $\vec{n}$ on it. Time rate of $\psi(t)$, when $\vec{B}$ is function of time and the boundary of surface is moving with arbitrary velocity $\vec{u}_\perp$, can be written as

$$\frac{d\psi}{dt} = \int \frac{\partial \vec{B}}{\partial t} \cdot \vec{n} dS + \oint \vec{B} \times \vec{u}_\perp \cdot dI.$$  \hspace{1cm} (28)

Taking into account $\vec{E} = -\vec{v}_\perp \times \vec{B}$ for an ideal MHD plasma leads us to flux conservation law:

$$\frac{d\psi}{dt} = \oint [\vec{v}_\perp \times \vec{B}] \cdot dI,$$  \hspace{1cm} (29)

which tells us that the flux through any arbitrary cross section is conserved when the cross sectional area of the flux tube moves with plasma ($\vec{u}_\perp = \vec{v}_\perp$). This has implication on plasma stability. If perturbation of plasma occurs, neighbouring plasma fluid elements are not allowed to tear or break into separate pieces. Since the field lines move with plasma in ideal MHD, the field-line topology must be preserved during any physical motion, which produces a strong constraint on the instabilities that can develop. If this kind of instability is excited, it is relative robust since it doesn’t depend on small plasma effects. If there is any even small resistivity, magnetic field lines can diffuse through the plasma and frozen-in field-line constraint is removed. This enables much more plasma motions and new instabilities can develop, occurring on a much slower time scale and with weaker effects.

7.2 Instability modes

There are several MHD instability modes which can be excited in plasma and need to be avoided in magnetic configurations. In general plasma stability can be improved by limiting the amount of pressure or toroidal current [3].
Interchange mode is a pressure driven mode which can be explained by considering a pure Z-pinch. Assume initially cylindrically shaped plasma is perturbed as shown in figure 9. Since the total current flowing in the cross section must remain constant all along the column, magnetic field ($B \sim 1/r$) at different locations satisfies $B_1 < B_2 < B_3$. Tension force on the surface of the plasma is also proportional to $F_T \sim 1/r$, which implies $F_1 < F_2 < F_3$. This makes the perturbation along the column to grow, since the force at $r_1$ is pointed radially outward and at $r_3$ radially inward. The perturbed force amplifies the initial perturbation and this corresponds to an unstable driving force. Another force distribution amplifying the perturbation in a twisting Z-pinch is caused by a difference in tension due to the different magnetic field amplitudes on regions of tight and gentle curvature as shown in figure 10.

![Figure 8: Schematic diagram of a Z-pinch interchange instability [3].](image)

![Figure 9: Schematic diagram of a twisting Z-pinch instability [3].](image)

Balloon modes can be explained on the example of linked mirrors shown in figure 11. MHD instabilities tend to concentrate plasma perturbations in the regions of unfavourable curvature, which corresponds to magnetic field lines that curve towards the surface of the plasma. Thus outward bulge on the surface put the plasma in a lower field region, which enhances the bulge. On the other side, when field lines curve away from the surface, the curvature is favourable. In that case an outward bulge places plasma in a region of higher magnetic field to its equilibrium position. Many magnetic configurations have both of curvatures and it is important that the average curvature is favourable.

In a twisted $\theta$-pinch configuration (figure 12) the instability is caused in a similar way as in pure Z-pinch and is called kink mode. However, the toroidal magnetic field lines become bent when the perturbation starts growing. Tension in the field lines tries to restraighten them and thus stabilize the perturbation. Instabilities can still occur if the toroidal field is to small.

![Figure 10: (a) Series of linked mirrors in equilibrium. (b) Favorable and unfavorable curvature regions with balloon perturbations [3].](image)

![Figure 11: Kink instability in a screw pinch with a large toroidal field [3].](image)
7.3 Conclusion

Macroscopic equilibrium and stability of plasma play a crucial role allowing fusion reactions to take place in continuous, steady state mode of operation inside the fusion magnetic device. Their analysis is based on a single-fluid model known as MHD. In general the equilibria of interest corresponds to toroidal configurations, in which the confined plasma is described by a set of nested, toroidal pressure surfaces. To achieve a toroidal equilibrium a solution of two problems is required. One is the radial pressure balance where the hot core of plasma tends to expand radially outward along the minor radius $r$. Several configurations are capable of providing radial pressure balance: the $\theta$-pinch, the $z$-pinch and the screw pinch. The second problem involves toroidal force balance as toroidal shape configuration generates forces that expand plasma outward along the major radius $R$. We must apply some additional forces to resist these outward forces. In a $\theta$-pinch there is no way to counteract these forces and toroidal equilibrium is not possible. For the macroscopic stability first the equilibrium needs to be found. However, several MHD instability modes can be excited in plasma and they need to be avoided, specially external MHD modes, which usually result in a catastrophic loss of plasma. Keeping the pressure and current low help the stability, but avoid the needs of a fusion reactor. What is more, frozen-in-field-line law keeps ideal MHD modes robust and unaffected by more subtle plasma physics phenomena.

References