Abstract

Kelvin-Helmholtz instability (KHI) is an instability at the interface between two parallel streams with different velocities and densities, with the heavier fluid at the bottom. We will take a look at two cases. In the first case we have a discontinuous profile of density and velocity and in the second the profile is continuous. We are interested under what conditions the flow instability occurs.
1 Introduction

Kelvin-Helmholtz instability (KHI) was first studied by Hermann von Helmholtz in 1868 and by William Thomson (Lord Kelvin) in 1871. It is a hydrodynamic instability in which incompressible and inviscid fluids are in relative and irrotational motion. In KHI as originally studied by Kelvin and Helmholtz, the velocity and density profiles are uniform in each fluid layer, but they are discontinuous at the (plane) interface between the two fluids. This discontinuity in the (tangential) velocity, i.e., the shear flow, induces vorticity at the interface; as a result, the interface becomes an unstable vortex sheet that rolls up into a spiral.[1] The heavier fluid parcels from lower denser fluid are lifted up and the lighter fluid parcels from lighter upper fluid are pushed down so overall we gain potential energy. The needed energy comes from the kinetic energy of the mean flow.

The name KHI is also used to describe the more general case where the variations of velocity and density are continuous across the interface.

KHI occurs not only in the atmosphere (Fig. 1) and ocean but also in motion of interstellar clouds [2, 3] and clumping in supernova remnants [4] in astrophysics, production of unstable shear layer and vortices in high energy density plasmas [5, 6], and quantized vortices in quantum fluids [7, 8].

Figure 1: Kelvin–Helmholtz instability in atmosphere. [9]
2 Instability of Discontinuously Stratified Parallel Flows

Figure 2 shows our two-dimensional model of heavier fluid ($\rho_2$) with velocity $\bar{u}_2$ on the bottom and lighter fluid ($\rho_1$) with velocity $\bar{u}_1$ at the top. Vertical and horizontal domains are infinite to prevent borders to have influence on development of instability. We deal with the problem on sufficiently small scales. The lateral scale must be smaller than effective Rosby radius so we don’t have to consider Coriolis force exerting influence on the flow. However, it must be large enough to ignore surface tension, molecular viscosity and density diffusion. By Kelvin’s circulation theorem, the perturbed flow must be irrotational in each layer because the motion develops from an irrotational basic flow of uniform velocity in each layer.

It’s a small scale phenomenon which saves us the need to use equations of motion in a rotating coordination frame thus simplifying their derivation.

\[ \nabla \cdot \mathbf{v} = 0. \]  

(1)

Because we have potential flow, the velocity is

\[ \mathbf{v} = \nabla \phi. \]  

(2)

Substitution into equation (1) gets us Laplace equations:

\[ \nabla^2 \phi = 0. \]  

(3)

Flow is decomposed into a basic state plus perturbations where index $j = 1, 2$ represents the variable of first and second layer:

\[ \phi_i = \bar{u}_j x + \phi'_j, \]  

(4)

where $\bar{u}_j$ represent the basic flow of uniform stream. Using equations (4) and substituting into equation (3) gives us

\[ \nabla^2 \phi'_j = 0, \]  

(5)

subject to
\[ \phi_1' \to 0 \text{ as } z \to \infty, \]
\[ \phi_2' \to 0 \text{ as } z \to -H. \]

(6)

For our purpose we can assume \( H \to \infty \) because depth is much bigger than wave amplitude.

At the interface we have kinematic boundary condition which requires that the fluid particle at the interface must move with the interface. So the vertical velocity just above and below interface must be equal. Considering particles just above the interface, this requires

\[ \frac{\partial \phi_j'}{\partial z} = \frac{d \zeta}{dt} = \frac{\partial \zeta}{\partial t} + \left( \bar{u}_j + u_j' \right) \frac{\partial \zeta}{\partial x} \quad \text{at } z = \zeta. \]

(7)

We expand the term on the left hand side of equation (7) in a Taylor series around \( z = 0 \):

\[ \left. \frac{\partial \phi}{\partial z} \right|_{z=\zeta} = \left. \frac{\partial \phi}{\partial z} \right|_{z=0} + \zeta \frac{\partial^2 \phi}{\partial z^2} + \cdot \cdot \cdot \approx \left. \frac{\partial \phi}{\partial z} \right|_{z=0}. \]

(8)

So for upper and lower layer the kinematic boundary conditions are:

\[ \frac{\partial \phi_j'}{\partial z} = \frac{\partial \zeta}{\partial t} + \bar{u}_j \frac{\partial \zeta}{\partial x} \quad \text{at } z = 0. \]

(9)

Flow dynamics is described by Navier-Stokes equation. For inviscid fluid the equation is

\[ \rho \frac{\partial v}{\partial t} + \rho (v \cdot \nabla) v = -\nabla p + \rho g. \]

(10)

Because our flow is irrotational the advection term can be written as

\[ (v \cdot \nabla) v = \frac{1}{2} \nabla v^2 - v \times \nabla \times v = \frac{1}{2} \nabla v^2, \]

as \( \nabla \times v \) is zero.

We rewrite gravity acceleration into a gradient of gravity potential \( g = -\nabla (gz) \) and we get

\[ \rho_j \frac{\partial \nabla \phi}{\partial t} + \nabla \left( \frac{1}{2} \rho_j (\nabla \phi)^2 + \rho_j g z \right) = -\nabla p_j. \]

(12)

\( \nabla \) can be written in front of the first term because density is constant and integrating in horizontal and vertical directions we get unsteady Bernoulli equations

\[ \rho_j \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho_j (\nabla \phi)^2 + \rho_j g z = -p_j + C_j, \]

(13)

with \( C_j \) as constant. We can calculate them using the initial state of flow before introducing the perturbation

\[ C_j = \bar{p}_j + \frac{1}{2} \rho_j \bar{u}_j^2. \]

(14)

The dynamic boundary condition at the interface means that the pressure must be continuous across the interface if surface tension is neglected. This requires \( \bar{p}_1 = \bar{p}_2 \) at \( z = 0 \) so we can write the equation (14) as

\[ \frac{1}{2} \rho_1 \bar{u}_1^2 - C_1 = \frac{1}{2} \rho_2 \bar{u}_2^2 - C_2. \]

(15)

In equation (13) we have a term \( (\nabla \phi)^2 \) and we substitute equation (4) into it and ignoring higher order terms we obtain:
\[(\nabla \phi)^2 = (\tilde{u} + \frac{\partial \phi'}{\partial x})^2 + \left(\frac{\partial \phi'}{\partial z}\right)^2 \approx \tilde{u}^2 + 2\tilde{u} \frac{\partial \phi'}{\partial x}.\]  

(16)

We use that result in equation (13)

\[\rho_j \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho_j \left(\tilde{u}_j^2 + 2\tilde{u}_j \frac{\partial \phi'}{\partial x} + p_j + \rho_j g \zeta = C_j,\]  

(17)

and requiring \(p_1 = p_2\) at \(z = \zeta\), we obtain the following condition at the interface:

\[C_1 - \rho_1 \frac{\partial \phi'}{\partial t} - \frac{1}{2} \rho_1 \left(\tilde{u}_1^2 + 2\tilde{u}_1 \frac{\partial \phi'}{\partial x}\right) - \rho_1 g \zeta = C_2 - \rho_2 \frac{\partial \phi'}{\partial t} - \frac{1}{2} \rho_2 \left(\tilde{u}_2^2 + 2\tilde{u}_2 \frac{\partial \phi'}{\partial x}\right) - \rho_2 g \zeta.\]  

(18)

By summing up equations (15) and (18) we get

\[\rho_1 \left[ \frac{\partial \phi'}{\partial t} + \tilde{u}_1 \frac{\partial \phi'}{\partial x} + g \zeta \right]_{z=0} = \rho_2 \left[ \frac{\partial \phi'}{\partial t} + \tilde{u}_2 \frac{\partial \phi'}{\partial x} + g \zeta \right]_{z=0}.\]  

(19)

The perturbation therefore satisfy equation (5) and conditions (6), (8), (9), and (19). We solve those equations with

\[\left(\zeta, \phi', \phi_2\right) = \left(\hat{\zeta}, \hat{\phi}_1, \hat{\phi}_2\right)e^{ik(x-ct)},\]  

(20)

where \(k\) is real, but \(c = c_r + ic_i\) is complex. The flow is unstable if there exists a positive \(c_i\).

Equation (5) needs solutions of the form

\[\hat{\phi}_1 = Ae^{-kz},\]

\[\hat{\phi}_2 = Be^{kz},\]

where solutions exponentially increasing from the interface are ignored because of condition (6). Substituting equation (20) into equations (8), (9), and (19) gives us three homogeneous linear algebraic equations with three unknowns \((A, B, \hat{\zeta})\):

The Bernoulli equation (19) is

\[iAk \rho_1 (\tilde{u}_1 - c) + iBk \rho_2 (\tilde{u}_2 + c) + g \hat{\zeta} (\rho_1 - \rho_2) = 0.\]  

(21)

The kinematic conditions (8) and (9) yield

\[A + i\hat{\zeta} (\tilde{u}_1 - c) = 0\]  

(22)

\[B + i\hat{\zeta} (\tilde{u}_2 + c) = 0.\]  

(23)

The three equations (21) to (23) have a non-trivial solution when \(D\) (eq. 26) equals zero: \(D = 0\).

\[Dx = 0\]  

(24)

where

\[x = [A, B, \hat{\zeta}]^T\]  

(25)

and
\[ D = \begin{bmatrix} ik\rho_1(\bar{u}_1 - c) & ik\rho_2(-\bar{u}_2 + c) & g(\rho_1 - \rho_2) \\ 1 & 0 & i(\bar{u}_1 - c) \\ 0 & 1 & i(-\bar{u}_2 + c) \end{bmatrix} \]  \tag{26}

That results in (27):

\[ c^2(\rho_1 + \rho_2) - 2c(\bar{u}_1\rho_1 + \bar{u}_2\rho_2) + \frac{g}{k}(\rho_1 - \rho_2) + (\bar{u}_1^2\rho_1 + \bar{u}_2^2\rho_2) = 0. \]  \tag{27}

This is a quadratic equation and the solution for \( c \) are

\[ c = \frac{\bar{u}_1\rho_1 + \bar{u}_2\rho_2}{\rho_1 + \rho_2} \pm \sqrt{\frac{(\bar{u}_1\rho_1 + \bar{u}_2\rho_2)^2}{(\rho_1 + \rho_2)^2} - \frac{\frac{g}{k}(\rho_1 - \rho_2) + \bar{u}_1^2\rho_1 + \bar{u}_2^2\rho_2}{\rho_1 + \rho_2}}. \]  \tag{28}

For instabilities to grow we need \( c \) to have imaginary part. The result under square root has to be negative otherwise the waves are stable.

\[ \frac{(\bar{u}_1\rho_1 + \bar{u}_2\rho_2)^2}{(\rho_1 + \rho_2)^2} < \frac{\frac{g}{k}(\rho_1 - \rho_2) + \bar{u}_1^2\rho_1 + \bar{u}_2^2\rho_2}{\rho_1 + \rho_2}. \]  \tag{29}

After rearrangement of terms we arrive at

\[ (\bar{u}_1 - \bar{u}_2)^2 > \frac{2g(\rho_2 - \rho_1)}{k\rho_1\rho_2}. \]  \tag{30}

We can see from equation (28) that for each growing solution there is a corresponding decaying solution.

Assuming \( \rho_1 \approx \rho_2 \approx \rho \) the right side of the equation simplifies in approximate condition (31):

\[ (\bar{u}_1 - \bar{u}_2)^2 > \frac{2g(\rho_2 - \rho_1)}{k\rho}. \]  \tag{31}

If \( \bar{u}_1 \neq \bar{u}_2 \), one can always find short enough wavelengths that satisfies our requirement for instability. If there is velocity shear we can say that the flow is always unstable to short waves.

The Kelvin–Helmholtz instability is caused by the destabilizing effect of shear, which overcomes the stabilizing effect of stratification. KHI can be generated in the tilting tank laboratory in which a dense (in Fig. 3 coloured) fluid underlies the lighter fluid. When the tank is tilted the denser fluid flows down and the lighter fluid flows up and the velocity shear generates instability as seen in Fig. 3 and Fig. 4.

![Figure 3: Kelvin–Helmholtz instability generated by tilting a horizontal channel containing two liquids of different densities. The lower layer is dyed. Mean flow in the lower layer is down the plane and that in the upper layer is up the plane. [11]](image)
At the start we assume flow in each layer is irrotational. However, there exists strong shear vorticity in the interface layer so the vorticity for the whole system does not equal zero. This shear vorticity transforms into circular vorticity which is described in more detail in the following paragraph.

The tendency of vorticity in a parallel shear layer to accumulate into evenly spaced maxima is the primary driver of KHI. Given an initial wavelike perturbation, mass conservation requires accelerated horizontal flow above the crests and below the troughs (5a). The resulting current anomalies advect vorticity toward the center of the figure. The vorticity concentration induces vertical velocity perturbations that amplify the original wave (5b), resulting in positive feedback and exponential growth of the perturbation. The most-amplified wavelength depends on the details of the initial profiles, but typically ranges from 6 to 11 times the initial transition-layer thickness. Stable stratification tends to slow the growth of the billows, but also to accelerate breaking once the billows reach sufficient amplitude to overturn.[13]

We can use alternative approach to the problem by considering potential and kinetic energy. We take into account only the layer next to interface in which mixing occurs. Because heavier

Figure 4: Development of a Kelvin–Helmholtz instability in the laboratory. The upper and faster moving layer is slightly less dense than the lower layer. At first, waves form and overturn in a two-dimensional fashion but, eventually, three-dimensional motions appear that lead to turbulence and complete the mixing. [12]

Figure 5: Schematic representation of the positive feedback that drives shear instability. (a) Vorticity accumulation due to horizontal advection. (b) Amplification of the initial wave by induced vertical motions. [13]
fluid parcels are raised and lighter fluid parcels are lowered the potential energy after mixing is raised. For how much is shown in equation (33). The average density is $\rho = (\rho_1 + \rho_2)/2$ and we count the height from where the mixing occurs.

$$\text{Figure 6: Mixing of a two-layer stratified fluid with velocity shear. Vertical axes are unbounded. Mixing occurs in the layer close to interface with the height of } \Delta H. \text{ As heavier fluid is raised and lighter fluid lowered so potential energy is increased. That movement is provided by a decrease in kinetic energy of the flow. [12]}$$

$$\text{PE gain} = \text{PE final} - \text{PE initial} =$$

$$= \int_0^{\Delta H} \rho g z dz - \left[ \int_0^{\Delta H} \rho_2 g z dz + \int_{\Delta H}^{\Delta H} \rho_1 g z dz \right] =$$

$$= \frac{1}{2} \rho g \Delta H^2 - \left[ \frac{1}{2} \rho_2 g \frac{\Delta H^2}{4} + \frac{1}{2} \rho_1 g \frac{3 \Delta H^2}{4} \right] =$$

$$= \frac{1}{8} (\rho_2 - \rho_1) g \Delta H^2 \quad (32)$$

The source for this potential energy increase has to come from somewhere and it comes from decrease of kinetic energy during mixing. Momentum is conserved as there are no external forces and using Bousinessq approximation we see that final velocity is the average of initial velocities $\bar{u} = (\bar{u}_1 + \bar{u}_2)/2$. Kinetic energy loss is

$$\text{KE loss} = \text{KE initial} - \text{KE final} =$$

$$= \int_0^{\Delta H} \frac{1}{2} \rho \bar{u}_2^2 dz + \int_{\Delta H}^{\Delta H} \frac{1}{2} \rho \bar{u}_1^2 dz - \int_0^{\Delta H} \frac{1}{2} \rho \bar{u}^2 dz$$

$$= \frac{1}{2} \rho \bar{u}_2^2 \Delta H + \frac{1}{2} \rho \bar{u}_1^2 \Delta H - \frac{1}{2} \rho \bar{u}^2 \Delta H$$

$$= \frac{1}{8} \rho (\bar{u}_1 - \bar{u}_2)^2 \Delta H. \quad (33)$$

There has to be bigger kinetic energy loss than potential energy gain for mixing to occur in the whole layer of $\Delta H$. Necessary condition is thus

$$g \Delta H (\rho_2 - \rho_1) \rho (\bar{u}_1 - \bar{u}_2)^2 < 1. \quad (34)$$

Rearranging the terms we see it is very similar to equation (31)

$$(\bar{u}_1 - \bar{u}_2)^2 > \frac{g \Delta H (\rho_2 - \rho_1)}{\rho}. \quad (35)$$

This is a condition for mixing to occur in the layer $\Delta H$. It is expected that the vertical extent $\Delta H/2$ scales like the wavelength of the longest unstable wave so criterion (31) becomes equality:
\[
\Delta H \sim \frac{1}{k_{\text{min}}} = \frac{\rho(\bar{u}_1 - \bar{u}_2)^2}{2g(\rho_2 - \rho_1)}.
\]

### 3 Instability of Continuously Stratified Parallel Flows

Kelvin and Helmholtz only studied discontinuously stratified flow but the more general case of continuously stratified flow is also commonly called the Kelvin-Helmholtz instability. Such flow was first studied by Taylor in 1915. He surmised that a gradient Richardson number must be less than 1/4 for instability. Prandtl, Goldstein, Richardson, Synge, and Chandrasekhar suggested values from 2 to 1/4 for the critical Richardson number. Miles (1961) was able to prove Taylor’s conjecture and Howard (1961) generalized Miles’ proof which is presented here.[10, 12]

We shall consider a two-dimensional \((x, z)\) inviscid and non-diffusive fluid with horizontal and vertical velocities \((u, w)\), dynamic pressure \(p\), and density anomaly \(\rho\). Two-dimensional disturbances are more unstable than three-dimensional which is proved by Squire’s theorem. Kelvin-Helmholtz is basically a two-dimensional phenomenon, only later in its development three-dimensional movements appear which leads to turbulent mixing.

Our basic state consists of a steady, sheared horizontal flow \([u = \bar{u}(z), w = 0]\) in a vertical density stratification \([\rho = \rho(z)]\). The accompanying pressure field \(p(z)\) obeys \(d\bar{p}/dz = -g\bar{\rho}(z)\). We add an infinitesimally small perturbation \((u = \bar{u} + u', w = w', p = \bar{p} + p', \rho = \bar{\rho} + \rho')\) and a following linearization of the momentum equations in \(x\) and \(z\) direction, continuity and energy equations become:

\[
\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + w \frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x},
\]

\[
\frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0},
\]

\[
\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0,
\]

\[
\frac{\partial \rho'}{\partial t} + \bar{u} \frac{\partial \rho'}{\partial x} + \rho_0 N^2 w' g \frac{\partial \psi}{\partial x} = 0.
\]

Defining a stream function through \(u' = \partial \psi/\partial z, w' = -\partial \psi/\partial x\) the continuity equation can be satisfied.

Equation (40) can be written as

\[
\frac{\partial \rho'}{\partial t} + \bar{u} \frac{\partial \rho'}{\partial x} - \frac{\rho_0 N^2 w'}{g} = 0,
\]

where we have defined \(N^2 = -(g/\rho_0)(\partial \bar{\rho}/\partial z)\) as the buoyancy frequency which is also known as Brunt–Väisälä frequency. It is the angular frequency at which a vertically displaced parcel will oscillate within a statically stable environment.

Substituting stream functions to equations (37), (38), and (41) gives us:

\[
\frac{\partial^2 \psi}{\partial z \partial t} + \frac{\partial \bar{\psi}}{\partial z} + \frac{\partial^2 \psi}{\partial x \partial z} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x},
\]

\[
\frac{\partial^2 \psi}{\partial x \partial t} + \frac{\partial \bar{\psi}}{\partial x} = -\frac{g \rho'}{\rho_0} - \frac{1}{\rho_0} \frac{\partial p'}{\partial z},
\]

\[
\frac{\partial \rho'}{\partial t} + \bar{u} \frac{\partial \rho'}{\partial x} + \frac{\rho_0 N^2 \partial \psi}{g} \frac{\partial \psi}{\partial x} = 0.
\]
Equations (42) is solved by assuming normal mode solutions of the form

$$[\rho', p', \psi'] = [\hat{\rho}, \hat{p}, \hat{\psi}]e^{ik(x-ct)}, \quad \text{(43)}$$

where quantities denoted by (\hat{\cdot}) are complex amplitudes. Because the flow is unbounded in \(x\), the wave number \(k\) must be real. The eigenvalue \(c = c_r + ic_i\) can be complex, and the solution is unstable if there exists a \(c_i > 0\), because then the exponent term is positive and grows with time.

Substituting normal modes into equations (42) we get

\begin{align*}
(u - c) \frac{\partial \hat{\psi}}{\partial z} - \frac{\partial u}{\partial z} \hat{\psi} &= -\frac{1}{\rho_0} \hat{\rho}, \quad \text{(44)} \\
k^2 (u - c) \hat{\psi} &= \frac{g \hat{\rho} \rho}{\rho_0} - \frac{1}{\rho_0} \frac{\partial \hat{\rho}}{\partial z}, \quad \text{(45)} \\
(u - c) \hat{\rho} + \frac{\rho_0 N^2}{g} \hat{\psi} &= 0. \quad \text{(46)}
\end{align*}

We want to reduce the above system of equations to single equation for \(\psi\). The pressure can be eliminated by taking the z-derivative of equation (44) and subtracting equation (45). The density can be eliminated by equation (46). Result is

\begin{equation}
(u - c) \left( \frac{\partial^2 \psi}{\partial z^2} - k^2 \psi \right) + \left( \frac{N^2}{u - c} - \frac{\partial^2 \bar{u}}{\partial z^2} \right) \psi = 0, \quad \text{(47)}
\end{equation}

which is Taylor–Goldstein equation and it governs the behaviour of perturbations in a stratified parallel flow.

At the top of the atmosphere and at the ground we impose vertical speed of zero. This requires \(\partial \psi/\partial x = ik \hat{\psi} \exp(ikx - ikct) = 0\) at the boundaries, which is possible only if

$$\psi(0) = \psi(H) = 0. \quad \text{(48)}$$

Note that the complex conjugate of the equation is also a valid equation because we can take the imaginary part of the equation, change the sign, and add to the real part of the equation. Because the Taylor–Goldstein equation does not involve any \(i\), a complex conjugate of the equation shows that if \(\hat{\psi}\) is an eigenfunction with eigenvalue \(c\) for some \(k\), then \(\hat{\psi}^*\) is a possible eigenfunction with eigenvalue \(c^*\) for the same \(k\). Therefore, to each eigenvalue with a positive \(c_i\) there is a corresponding eigenvalue with a negative \(c_i\). In other words, to each growing mode there is a corresponding decaying mode. A non-zero \(c_i\) therefore ensures instability.[10]

### 3.1 Richardson Number Criterion

It’s impossible to solve equations (47) and (48) in the general case for an arbitrary shear flow \(\bar{u}(z)\) so we will derive integral constraints. We replace \(\psi\) with

$$\psi = \sqrt{u - c} \phi. \quad \text{(49)}$$

Boundary (47) and and boundary conditions (48) become

\begin{equation}
\frac{\partial}{\partial z} \left[ (u - c) \frac{\partial \phi}{\partial z} \right] - \left[ k^2 (u - c) + \frac{1}{2} \frac{\partial^2 \bar{u}}{\partial z^2} + \frac{1}{4} \left( \frac{\partial \bar{u}}{\partial z} \right)^2 - N^2 \right] \phi = 0, \quad \text{(50)}
\end{equation}

9
\[ \phi(0) = \phi(H) = 0. \]  

(51)

Multiplying equation (50) by the complex conjugate \( \phi^* \), integrating over the vertical scale, and using conditions (51), we get:

\[
\int_0^H \left[ N^2 - \frac{1}{4} \left( \frac{\partial \bar{u}}{\partial z} \right)^2 \right] \frac{\phi^2}{(\bar{u} - c)} \, dz = \int_0^H (\bar{u} - c) \left( \left| \frac{\partial \phi}{\partial z} \right|^2 + k^2 |\phi|^2 \right) \, dz + \frac{1}{2} \int_0^H \frac{\partial^2 \bar{u}}{\partial z^2} \, dz,
\]

(52)

where vertical bars denote the absolute value of complex quantities. The imaginary part of this expression is

\[
c_i \int_0^H \left[ N^2 - \frac{1}{4} \left( \frac{\partial \bar{u}}{\partial z} \right)^2 \right] \frac{\phi^2}{|\bar{u} - c|^2} \, dz = -c_i \int_0^H \left( \left| \frac{\partial \phi}{\partial z} \right|^2 + k^2 |\phi|^2 \right) \, dz,
\]

(53)

where \( c_i \) is the imaginary part of \( c \). If the flow is such that \( N^2 > \frac{1}{4} (d\bar{u}/dz)^2 \) everywhere, then the prior equation states that \( c_i \) times a positive quantity equals \( c_i \) times a negative quantity. This is impossible and requires that \( c_i = 0 \). We define Richardson number as

\[
R_i = \frac{N^2}{(\partial \bar{u}/\partial z)^2} = \frac{N^2}{M^2}.
\]

(54)

Linear stability is guaranteed if the inequality

\[
R_i > \frac{1}{4}
\]

(55)

is satisfied everywhere in the flow.

This does not mean that flow is unstable if we have \( R_i < \frac{1}{4} \) somewhere in the flow. It is a necessary but not sufficient condition for instability.

### 4 Conclusion

This paper shows mathematical derivation of equations needed for understanding Kelvin-Helmholtz instability. We see that flow is always unstable for small enough wavelengths (while ignoring surface tension, viscosity and diffusion) if there is velocity shear present in discontinuously stratified flow.

In continuously stratified flow we define Richardson number and if it is bigger than 1/4 the flow is stable. For flow to be unstable \( R_i \) has to be lower than 1/4 but it does not guarantee instability.

### References


