Abstract

We review the Grossmann-Lohse theory of global properties for Rayleigh and Prandtl number dependence on Nusselt number and Reynolds number in the turbulent Rayleigh-Bénard convection. As we have heard very little about this kind of hydrodynamic instability so far in the course of our studies, we shall derive the equations of motion using the Boussinesq approximation. The goal of the first part is to show where do some important dimensionless numbers in hydrodynamics come from and what do they represent.

The main part is dedicated to the Grossmann-Lohse theory which is one of the first successful theories to describe the Ra and Pr dependence of Nu and Re over the wide parameter range. The theory is based on the decomposition of the energy dissipation rate into boundary layer and bulk contribution.
Introduction

The observation of thermal instability goes back a century to 1882, when James Thompson, the older brother of better known Lord Kelvin, had first described the convective instability. It was later Bénard in 1900 who has discovered in the first quantitative experiments the making of regular hexagons in convection cells in silicone oil (Figure 1). Impressed by these results, Rayleigh in 1916 formulated a theory of convection instability of a layer of fluid between horizontal planes. He showed that the instability would occur only when the adverse temperature gradient was large enough [1].

Figure 1: An example of Bénard convection cells in hexagonal shape [2].

1 Reyleigh-Bénard convection

Rayleigh-Bénard convection is the buoyancy driven flow where the fluid is heated from below and cooled from above. When the temperature difference across the layers is large enough, the stabilizing effects of viscosity and thermal conductivity are overcome by the destabilizing buoyancy, and an overtuning instability is shown as thermal convection. This instability can be distinguished from free convection. For example imagine a hot vertical plate, for which hydrostatic equilibrium is impossible [3], [1].

Reyleigh-Benard convection is a classical problem in fluid dynamics. It has played an important role in the development of stability theory in hydrodynamics and it made the basis in the study of pattern formation and spatial-temporal chaos.

Thermally driven flows play an important role in a variety of physical problems. Thermal convection can be found in the atmosphere [4], in the oceans [5], process technology, in metal-production processes [6], and even in buildings [7]. It has also been seen in geophysical and
astrophysical context, such as convection of Earth’s mantle [8], in the Earth’s outer core [9], in stars including our Sun [10], and in generation and reversal of the Earth’s magnetic field [11]. As we see thermal convection is a very important physical phenomenon as it is found in all branches of physics [3].

2 The equations of motion

The Reyleigh-Bénard convection problem can be described by the following question: For a given fluid in a closed container of height $L$ heated from below and cooled from above, what are the flow properties inside the container, and what is the heat transfer from bottom to top. Here we will assume spatially and temporally constant temperatures at the bottom and top.

In the notation of Cartesian tensors with position vectors $\mathbf{x} = x_j$ and velocity $\mathbf{u} = u_j$ ($j = 1, 2, 3$), the equations written in the standard notation where we sum over repeated indices are as follows

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} = 0.$$  \hfill (1)

This is the equation of continuity where $\rho$ is the density of the fluid. The equation of motion is the Navier-Stokes equation:

$$\rho \frac{D u_i}{D t} = -g\rho \delta_{i3} + \frac{\partial \sigma_{ij}}{\partial x_j},$$  \hfill (2)

where $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$, the $x_3$-axis is the upward vertical, $g$ is the gravitational acceleration, and the stress tensor is given by

$$\sigma_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij},$$  \hfill (3)

where $p$ is the pressure, $\mu$ is the coefficient of dynamic viscosity of the fluid, and $\lambda$ is the coefficient of bulk viscosity.

Next is the equation of energy, or of heat conduction which we obtain from [1], [3], [12]

$$\rho \frac{DE}{Dt} = -\frac{\partial}{\partial x_j} \left( k \frac{\partial T}{\partial x_j} \right) - p \frac{\partial u_j}{\partial x_j} + \Phi,$$  \hfill (4)

where $E$ is the internal energy per unit mass of the fluid, $k$ is the thermal conductivity, and $\Phi$ is the rate of viscous dissipation per unit volume of fluid which is given by

$$\Phi = \frac{1}{2} \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 + \left( \lambda - \frac{2}{3} \mu \right) \left( \frac{\partial u_k}{\partial x_k} \right)^2.$$  \hfill (5)

For a perfect gas $E = c_v T$, and for a liquid $E = c T$, where $c_v$ is the specific heat at constant volume and $c$ is the specific heat at constant pressure [1], [3].

2.1 The Boussinesq equations

To upper equations of motion, Rayleigh in 1916 applied the Boussinesq approximation. The main idea behind this approximation is that there are flows in which the temperature varies only a little, and therefore the density varies only a little, but the buoyancy still drives the
motion. Then we can neglect the variation of density everywhere except in the buoyancy. For small differences in temperature we can therefore write

$$\rho = \rho_0 \{1 - \alpha(T - T_0)\}, \quad (6)$$

where $\alpha$ is the coefficient of thermal cubical expansion, $\rho_0$ is the density of the fluid at the temperature $T_0$ at the bottom of the layer. We already know that for a perfect gas $\alpha = 1/T_0 \approx 3 \times 10^{-3}K^{-1}$, and for a typical liquid used in experiments $\alpha \approx 5 \times 10^{-4}K^{-1}$. Therefore if $T_0 - T \approx 10K$ or less, then

$$\frac{\rho - \rho_0}{\rho_0} = \alpha(T_0 - T) \ll 1, \quad (7)$$

but the buoyancy $g(\rho - \rho_0)$ is still of the same order of magnitude as the inertia, acceleration or viscous stresses of the fluid and is therefore not negligible [1], [3].

For most real fluids $\frac{d\mu/\mu}{dT}$, $\frac{dk/k}{dT}$, $\frac{dc/c}{dT} \leq \alpha$, so $\mu$, $k$, and $c$ or $c_v$ can be treated as constants in the Boussinesq approximation. The coefficient of bulk viscosity $\lambda$ is neglected, as it only arises as a factor of $\partial u_j/\partial x_j$, which is of order $\alpha$.

To sum up, one approximates all the thermodynamic variables as constants except for the pressure and temperature. Density is a non-constant variable only when multiplied with $g$ [1].

In the continuity equation the differentials of density are of order $\alpha$, so the Boussinesq approximation gives (where we of course sum over repeated indices)

$$\frac{\partial u_i}{\partial x_j} = 0, \quad (8)$$

which is an approximation for incompressible fluid. If we threat $\rho$ and $\mu$ as constants in every term except for the buoyancy, the Navier-Stokes equations become

$$\frac{D u_i}{D t} = -\frac{\partial}{\partial x_i} \left( \frac{p}{\rho_0} + gz \right) - \alpha g(T_0 - T)\delta_{i3} + \nu \Delta u_i, \quad (9)$$

where $\Delta$ is the Laplacian operator, given by $\Delta = \sum_j \partial^2/\partial x_j^2$, and $\nu$ is kinematic viscosity [1]. The kinematic viscosity is defined as ratio of dynamic viscosity and the density of the fluid. Sometimes it is referred to as diffusivity of momentum as it is analogous to diffusivity of heat.

In the energy we can neglect the viscous dissipation $\Phi$ as the ratio of the rate of production of heat by internal friction to the rate of transfer of heat is

$$\frac{\Phi}{\rho D(cT)/Dt} \approx \frac{\mu U_2 L^{-1}}{\rho_0 c \delta TL^{-1}} = \frac{\nu U}{c \delta TL}, \quad (10)$$

where $U$ represents velocity scale of the flow, $L$ a length scale, and $\delta T = (T_0 - T_1)$ a scale of temperature difference, where $T_1$ is the temperature of the top layer of thickness $L$. For a typical gas $\nu/c_v \approx 10^{-8}s$ K and for a typical liquid $\nu/c \approx 10^{-9}s$ K. If we assume that $U/(\delta TL)$ is not very large, the upper ratio is very small for both gases and liquids. Therefore we can neglect $\Phi$ in our equations [1], [3].

Now note that heating due to compression is given by

$$-p \frac{\partial u_j}{\partial x_j} = \frac{p}{\rho} = \alpha p \frac{DT}{Dt} \quad (11)$$
where for a perfect gas, \( p = (c_p - c_v)\rho T \) and \( \alpha = 1/T \). Now it follows

\[
\rho \frac{DE}{Dt} + p\frac{u_j}{x_j} \approx c_p\rho \frac{DT}{Dt}
\]

One might expect from the approximation for the incompressible fluid that heating due to compression is negligible but this is not the case. For liquids, however, the heating is negligible at normal pressures. The reason for the difference between liquids and gases is mainly that the heating due to compression is normally an order of magnitude less for a liquid than for a gas. The heat transfer is also proportional to the density of the fluid, and a typical fluid is \( 10^3 \) times more dense than a typical gas [1], [3].

Now using all these approximations and the approximation for incompressible fluid we get

\[
\frac{DT}{Dt} = \kappa \Delta T,
\]

where \( \kappa \) is thermal diffusivity with \( \kappa = k/(\rho_0 c_p) \) for perfect gas and \( \kappa = k/(\rho_0 c) \) for a liquid [1].

Equations (8), (10) and (14) are called Boussinesq equations and describe the motion of Boussinesq fluid.

### 2.2 Important dimensionless numbers

Now we will focus on the boundary conditions of the Boussinesq equation. Let the two horizontal planes have equations \( z^* = 0 \) and \( z^* = L \), where the temperatures are \( T_0 \) and \( T_1 \), respectively. Here we denote the dimensional variable with an asterisk as we shall soon introduce dimensionless variables.

The equations of motion for velocity, temperature, and pressure are therefore for the basic state (that is a state where the gradient is a linear function and no velocity is observed) given by \( u^* = 0 \), \( T^* = T_0 - \beta z^* \), \( P^* = p_0 - g \rho_0 (z^* + \frac{1}{2} \alpha \beta z^2) \) for \( 0 \leq z^* \leq L \). Let us repeat what the parameters are; \( \alpha \) is coefficient of thermal cubical expansion, \( \rho_0 \) the density of the fluid at bottom of the layer, and \( \beta = (T_0 - T_1)/L \) is the basic temperature gradient. We assume that an instability can only occur when \( T_0 > T_1 \), meaning the temperature gradient \( \beta > 0 \).

If we linearize the Boussinesq equations for small perturbations \( u'_s, T'_s, \) and \( p'_s \) with \( u_s = u'_s(x_s, t_s), T_s = T_s(z) + T'_s(x_s, t_s), p_s = P_s(z) + p'_s(x_s t_s) \) we get

\[
\nabla_s \cdot u'_s = 0, \quad (14)
\]

\[
\frac{\partial u'_s}{\partial t_s} = -\frac{1}{\rho_0} \nabla_s p'_s + \alpha g T'_s k + \nu \Delta_s u'_s, \quad (15)
\]

\[
\frac{\partial T'_s}{\partial t_s} = \beta w'_s + \kappa \Delta_s T'_s, \quad (16)
\]

where \( w'_s \) is the velocity in z-direction, and \( \mathbf{n} \) is the normal to the lower plate [1], [3].

Now it is convenient to define dimensionless variables using \( L \) for the scale of length, \( L^2/\kappa \) for scales of time, and \( \beta L = T_0 - T_1 \) for scales of temperature difference;

\[
x = \frac{x_s}{L}, \quad t = \frac{\kappa t_s}{L^2}, \quad (17)
\]

\[
u = \frac{L \nu_s}{\kappa}, \quad T = \frac{T'_s}{\beta L}, \quad p = \frac{L^2 p'_s}{\rho_0 \kappa^2}, \quad (18)
\]
The linearized stability equations (14-16) then become

\[ \nabla \cdot \mathbf{u} = 0, \]  
\[ \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \text{Ra} \ \text{Pr} \ \mathbf{n} + \text{Pr} \ \Delta \mathbf{u}, \]  
\[ \frac{\partial T}{\partial t} = w + \Delta T, \]

where the dimensionless \textit{Rayleigh number} is given by

\[ \text{Ra} = \frac{g \alpha \beta L^4}{\kappa \nu}, \]  

One should note that the Ra is positive when \( T_0 > T_1 \) and it represents the characteristic ratio of buoyancy to the viscous forces [1].

\textit{Prandtl number} is given by

\[ \text{Pr} = \frac{\nu}{\kappa}. \]  

Prandtl number is an intrinsic property of the fluid, not of the flow. It is a measurement of the ratio of the rates of molecular diffusion of momentum and heat.

When a system is imposed by Ra, the heat flux \( H \) appears from bottom to top. We can accordingly define dimensionless heat flux called \textit{Nusselt number} as

\[ \text{Nu} = \frac{H}{k \Delta T L^{-1}}, \]  

where \( \Delta T \) is the temperature difference between top and bottom plate.

Furthermore we define \textit{Reynolds number} as

\[ \text{Re} = \frac{U}{\nu L^{-1}}, \]  

where \( U \) represents velocity scale of the flow. As the Reynolds number has already been vastly discussed in the first cycle of faculty courses, we will skip the description of its derivation and importance [3].

We shall stop here and will not exploit more about the solutions of the equations (27-29) as this is not the main topic of this seminar.

### 3 Theories of global properties: \text{Nu}(\text{Ra}, \text{Pr}) and \text{Re} (\text{Ra}, \text{Pr})

The key question is to ask ourselves how do Nu and Re depend on Ra and Pr. Older theories predict power laws for dependences of Nu and Re on Ra and Pr

\[ \text{Nu} \sim \text{Ra}^{\gamma_{\text{Nu}}} \text{Pr}^{\alpha_{\text{Nu}}}, \]  
\[ \text{Re} \sim \text{Ra}^{\gamma_{\text{Re}}} \text{Pr}^{\alpha_{\text{Re}}}, \]

and are mostly obtained from approximations at certain regimes of the variables. These older theories are all of limited range and they do not offer a unifying view, accounting for all data [3], [13].

We will now describe the Grossmann-Lohse theory which offers a unifying view over a wide range of parameter.
3.1 Grossmann-Lohse theory

The main approach of the Grossmann-Lohse theory is to decompose the kinetic energy dissipation rate $\epsilon_u$ and the thermal dissipation rate $\epsilon_T$ into their boundary layer (BL) and bulk contributions (see Figure 2) [12], [14], [15], [3]:

$$\epsilon_u = \epsilon_{u, BL} + \epsilon_{u, bulk},$$  \hspace{1cm} (28)

$$\epsilon_T = \epsilon_{T, BL} + \epsilon_{T, bulk}$$  \hspace{1cm} (29)

![Figure 2: Boundary-bulk partition. We see the sketch splitting of the (a) kinetic and (b) thermal dissipation rates on which the Grossmann-Lohse theory is based. In both figures the large scale convection which roll with typical velocity amplitude $U$ is sketched. The typical width of the kinetic BL is $\lambda_u$, whereas the typical thermal BL thicknesses and the plume thicknesses are $\lambda_T$. Outside the BL/plume region we see background (bg) / bulk flow [3].](image)

The exact relations assuming statistical stationarity for the left-hand sides following [13], [14] are

$$\epsilon_u = \langle \nu [\partial_i u_j(x, t)]^2 \rangle_V = \frac{\nu^3}{L^4}(\text{Nu} - 1) \text{Ra Pr}^{-2},$$  \hspace{1cm} (30)

$$\epsilon_T = \langle \kappa [\partial_i T(x, t)]^2 \rangle_V = \kappa \frac{\Delta T^2}{L^2} \text{Nu}$$  \hspace{1cm} (31)

As the next description of the Grossmann-Lohse theory is a bit difficult to understand, we will divide it into separate parts so we will easily distinguish between different important steps in the understanding of the theory.

1. The local dissipation rates of both kinematic and thermal contribution are modelled in the terms of $U$, $\Delta T$, $L$, and the widths $\lambda_u$, $\lambda_T$ of the kinetic and thermal boundary layers, respectively [14]. For the thickness of the thermal boundary layer we assume $\lambda_T = L/(2\text{Nu})$ and for the kinetic width $\lambda_u = L/(4\sqrt{\text{Re}})$, same as for Blasious type layers [12], [14], [16]. For very large Ra the laminar boundary layer will become turbulent and $\lambda_u$ will show stronger Re dependence. One should also note that while thermal boundary layer occurs only at the top and the bottom plate, the kinetic boundary layer occurs at all walls of the cell [14].

2. The modeling of both dissipation rates in equations (28) and (29) is obeying the Boussinesq equations. It depends on which contribution is dominant (BL or bulk),
the solutions for relation for $\text{Nu}$, $\text{Re}$ vs $\text{Ra}$, $\text{Pr}$ differ, and therefore defining different main regimes; see Figure 3 [14], [3]. In regime I both $\epsilon_u$ and $\epsilon_T$ are dominated by their boundary contributions; regime II occurs when $\epsilon_T$ is dominated by $\epsilon_{u,\text{BL}}$ and $\epsilon_u$ is dominated by $\epsilon_{T,\text{BL}}$; regime III comes into consideration when $\epsilon_T$ is dominated by $\epsilon_{u,\text{BL}}$ and $\epsilon_T$ is dominated by $\epsilon_{T,\text{bulk}}$; regime IV occurs when both $\epsilon_T$ and $\epsilon_u$ are bulk
dominated.

3. The BL and bulk thermal dissipation rates therefore depend on whether the kinetic BL of thickness $\lambda_u$ is within the thermal BL of thickness $\lambda_T$ (meaning $\lambda_u < \lambda_T$, that is for small Pr) or vice versa (meaning $\lambda_T < \lambda_u$, that is for large Pr); see Figure 4. The boundary case when $\lambda_u = \lambda_T$, which corresponds to $\text{Nu} = 2\sqrt{\text{Re}}$, cuts the phase diagram into two parts; lower with small Pr is labelled by "l", upper with large Pr with "u" (see Figure 2 and Figure 3 ) [14].

Figure 3: Phase diagram in the $\text{Ra}$-$\text{Pr}$ plane. The tiny regime to the right of regime I is regime III. The dashed line is $\lambda_u = \lambda_T$. The shaded regime for large Pr is where $\text{Re} \leq 50$, and in the shaded regime for low Pr we have $\text{Nu} = 1$. The dotted line indicates the non-normal-nonlinear onset of turbulence in the BL shear flow which we will not discuss here [12].

Figure 4: Sketch of the boundary layers with large-scale velocity $U$, (a) for low Pr where $\lambda_u < \lambda_T$ and (b) for large Pr where $\lambda_u > \lambda_T$ [12].

4. We first consider $\lambda_u < \lambda_T$ (small Pr, regime "l"). Assuming large-scale velocity $U$
stirring the bulk, one can estimate the bulk dissipation rate $\epsilon_{u,\text{bulk}}$ by balancing the
dissipation with the large-scale convective term in the energy equation. With the same analogy we can also get $\epsilon_{T,\text{bulk}}$. It follows from [12], [14]

$$\epsilon_{u,\text{bulk}} \sim \frac{U^3}{L} \sim \frac{\nu^3}{L^4} \text{Re}^3,$$

(32)

$$\epsilon_{u,\text{BL}} \sim \frac{U^2 \lambda_u}{\lambda_u^2 L} \sim \frac{\nu^3}{L^4} \text{Re}^3,$$

(33)

$$\epsilon_{T,\text{bulk}} \sim \frac{U \Delta T^2}{L} \sim \kappa \frac{\Delta T^2}{L^2} \text{Re} \text{ Pr},$$

(34)

$$\epsilon_{T,\text{BL}} \sim \kappa \frac{\Delta T^2}{L^2} (\text{Re} \text{ Pr})^{1/2},$$

(35)

5. If we consider larger Pr (so we go from low Pr numbers in step 4 to medium sizes Pr), then the kinetic boundary layer will eventually exceed the thermal one, $\lambda_u > \lambda_T$, meaning the upper range "$u$" (see Figure 3). The relevant velocity at the edge between thermal BL and the thermal bulk is now less than $U$, within our approximation about $U \lambda_T/\lambda_u$; see Figure 4. For the transition from $\lambda_u$ being smaller to being larger than $\lambda_T$ we introduce a new function $f(x) = (1 + x^n)^{-1/n}$ of the variable $x = \lambda_u/\lambda_T = \text{Nu}/(2\sqrt{\text{Re}})$ [14], [3]. The function $f$ equals 1 in the lower range (called "l", meaning small in Pr) and $\approx 1/x$ in the upper range (called "u", meaning large in Pr). The relevant velocity then becomes $U f(\lambda_u/\lambda_T)$. Grossmann and Lohse [12], [14] took $n = 4$ to characterize the sharpness of the transition. We should note that this function is (within the theory) universal for all fluids. The correspondence to the experiments shows that the choice of function is justified. Using this correction we get

$$\epsilon_{T,\text{bulk}} \sim \kappa \frac{\Delta T^2}{L^2} \text{Re} \text{ Pr} f \left( \frac{\text{Nu}/(2\sqrt{\text{Re}})}{(\lambda_u/\lambda_T)^n} \right),$$

(36)

$$\epsilon_{T,\text{BL}} \sim \kappa \frac{\Delta T^2}{L^2} \left[ \text{Re} \text{ Pr} f \left( \frac{\text{Nu}/(2\sqrt{\text{Re}})}{(\lambda_u/\lambda_T)^n} \right) \right]^{1/2},$$

(37)

while $\epsilon_{u,\text{bulk}}$ and $\epsilon_{u,\text{BL}}$ are still given by equations (32-33). Using those equations together with equations (30-31) we get Ra, Pr dependences of Nu, Re in the upper regime "$u$".

6. Let us now see the theory for very large Prandtl numbers. In step 4 we have small Pr numbers, in step 5 medium sized Pr numbers and now the extension to very large Prandtl numbers. \Rightarrow As Ra_c (critical Ra number when instability occurs) is independent of Pr, the wind still exists for $\text{Ra} > \text{Ra}_c = 1708$ [14]. But for large scale convection Re slows down with increasing Pr. It eventually becomes laminar throughout the cell [14]. $\lambda_u$ in the equation $\lambda_u = L/(4\sqrt{\text{Re}})$ does no longer increase with decreasing Re, but will close up to a value of order $L$. We shall call the corresponding number $\text{Re}_c$ and $\lambda_T = L/(4\sqrt{\text{Re}_c})$ in the regime of very large Pr. As the transition for Re after $\text{Re}_c$ is not smooth, we use the crossover function $g(x) = x(1 + x^n)^{-1/n}$ of the variable $x_L = \lambda_u(\text{Re}/\lambda_u(\text{Re}_c)) = \sqrt{\text{Re}_c/\text{Re}}$, and here also $n = 4$ [14], [3]. Below the transition ($x_L$ small) function $g$ increases linearly $g(x_L) = x_L$, and equals 1 at very large regimes of Pr with $\text{Re} \leq \text{Re}_c$. All this put together means that we have to replace $\lambda_u$ with $g(x_L)\lambda_u(\text{Re}_c)$ [14].
In our case equation (32) still holds even at very large Pr regime as $\epsilon_u$, bulk does not contribute much to $\epsilon_u$ due to large extension of the kinetic boundary layers. Nevertheless the equations (33-35) have to be generalized

$$\epsilon_u,_{BL} \sim \nu^3 \frac{\text{Re}^2}{g(\sqrt{\text{Re}_c/\text{Re}})}; \quad (38)$$

$$\epsilon_{T,\text{bulk}} \sim \kappa \frac{\Delta T^2 \text{Pr} \text{Re}}{L^2 \text{Nu}} f \left[ \frac{\text{Nu}}{2\sqrt{\text{Re}_c}} g \left( \sqrt{\frac{\text{Re}_c}{\text{Re}}} \right) \right], \quad (39)$$

where equation (39) simplifies for large enough Ra or very large Pr (meaning large f argument or large g argument, respectively) to

$$\epsilon_{T,\text{bulk}} \sim \kappa \frac{\Delta T^2 \text{Pr} \text{Re}}{L^2 \text{Nu}} \quad (40)$$

For example how to obtain power laws describing heat flux and velocity, one inserts the equations (38) and (39) into the right side of equations (38) and (39) in the regime called III$_\infty$ for large enough Ra and Pr [12], [14]

$$\text{Nu} \sim \text{Ra}^{1/3} \text{Pr}^0; \quad \text{Re} \sim \text{Ra}^{2/3} \text{Pr}^{-1}. \quad (41)$$

The generalized $\epsilon_{T,\text{BL}}$ (equation 37) beyond I$_l$ (relevant for medium Ra) stays the same as its derivation did not involve $\lambda_u$ and function $f = 1$.

7. Putting all the generalized equations in equations (30) and (31), Grossmann and Lohse [14], [3] obtained the results of the Grossmann-Lohse theory

$$\text{Nu} \text{ Ra} \text{ Pr}^{-2} = c_1 \frac{\text{Re}^2}{g(\sqrt{\text{Re}_c/\text{Re}})} + c_2 \text{Re}^3, \quad (42)$$

$$\text{Nu} = c_3 \text{Re}^{1/2} \text{Pr}^{1/2} \left\{ f \left[ \frac{\text{Nu}}{2\sqrt{\text{Re}_c}} g \left( \sqrt{\frac{\text{Re}_c}{\text{Re}}} \right) \right] \right\}^{1/2} + c_4 \text{Pr} \text{Re} f \left[ \frac{\text{Nu}}{2\sqrt{\text{Re}_c}} g \left( \sqrt{\frac{\text{Re}_c}{\text{Re}}} \right) \right], \quad (43)$$

Some dimensionless prefactors were added to complete the modelling of the dissipation rates. They were adopted by nonlinear fit from a set of experimental data points for Nu(Ra,Pr) and the results for aspect ratio $\Gamma = 1$ were $c_1 = 120$, $c_2 = 74$, $c_3 = 0.89$, $c_4 = 0.048$, and $\text{Re}_c = 0.28$ [17], [14]. We should note that $c_i$ and $\text{Re}_c$ slightly depend on the aspect ratio and are not universal.

All limiting, pure scaling regimes derived from equations (42) and (43) can be seen in Table 1 where the approximations for typical sizes of variables (Pr and Ra) at each regimes were made and typical sizes of arguments in functions $f$ and $g$ were used. The phase diagram has been drawn in Figure 3. One has to note that even though lines indicate transitions have been made, the crossovers are nothing but sharp.

The success of the theory is of course in the correspondence to the actual experimental data. We can see on Figure 5 data together with GL prediction for Re in regime I. The agreement of the slope is very good. For more information see Figure 5.
Table 1: The pure power laws for Nu and Re. The prefactors $c_i$ are based on the values given below equation (43) [14].

<table>
<thead>
<tr>
<th>Regime</th>
<th>Dominance of</th>
<th>BLs</th>
<th>Nu</th>
<th>Re</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_l$</td>
<td>$\epsilon_u, BL, \epsilon_{uT}, BL$</td>
<td>$\lambda_u &lt; \lambda_T$</td>
<td>$0.22Ra^{1/4}Pr^{1/8}$</td>
<td>$0.063Ra^{1/2}Pr^{-3/4}$</td>
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<tr>
<td>$I_u$</td>
<td>$\epsilon_u, BL, \epsilon_{uT}, BL$</td>
<td>$\lambda_u &lt; \lambda_T$</td>
<td>$0.31Ra^{1/4}Pr^{-1/12}$</td>
<td>$0.073Ra^{1/2}Pr^{-5/6}$</td>
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<tr>
<td>$II_l$</td>
<td>$\epsilon_u, BL, \epsilon_{uT}, BL$</td>
<td>$\lambda_u &lt; \lambda_T$</td>
<td>$0.37Ra^{1/5}Pr^{1/5}$</td>
<td>$0.17Ra^{2/3}Pr^{-3/5}$</td>
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<tr>
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<td>$\epsilon_u, BL, \epsilon_{uT}, BL$</td>
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<tr>
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<td>$\epsilon_u, BL, \epsilon_{uT}, BL$</td>
<td>$\lambda_u &gt; \lambda_T$</td>
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<td>$III_\infty$</td>
<td>$\frac{L}{4\sqrt{Re_c}} &gt; \lambda_T$</td>
<td>$\lambda_u &gt; \lambda_T$</td>
<td>$0.027Ra^{1/3}$</td>
<td>$0.015Ra^{2/3}Pr^{-1}$</td>
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<tr>
<td>$IV_l$</td>
<td>$\epsilon_u, BL, \epsilon_{uT}, BL$</td>
<td>$\lambda_u &lt; \lambda_T$</td>
<td>$0.0012Ra^{1/2}Pr^{1/2}$</td>
<td>$0.025Ra^{1/2}Pr^{-1/2}$</td>
</tr>
<tr>
<td>$IV_u$</td>
<td>$\epsilon_u, BL, \epsilon_{uT}, BL$</td>
<td>$\lambda_u &gt; \lambda_T$</td>
<td>$0.050Ra^{1/3}$</td>
<td>$0.088Ra^{4/9}Pr^{-2/3}$</td>
</tr>
</tbody>
</table>

On Figure 6 one can see the functional dependence of Ra over a larger area on Nu. The agreement with the GL theory is again very good, for larger Ra some deviations occur. The theory is much better for data that is not corrected for finite plate conductivity as our approximation did not include it. See Figure 6.

Figure 5: Data for Raynolds numbers as a function of Ra and Pr from [18]. (a) shows $Re/Pr^{-3/4}$ vs Ra. The expected slope (in regime $I_l$) is $1/2$, the linear regression fit (solid line) gives $0.492 \pm 0.002$. (b) shows $Re/Ra^{1/2}$ vs Pr. The extended slope is $-3/4$ for $Pr \leq 2$ (regime $I_l$) and $-5/6$ for $Pr \geq 2$ (regime $I_u$). The linear regression fit (dashed line) gives $-0.77 \pm 0.01$. The agreement of the prefactor is also excellent. According to the theory, following table 1 we get $\log_{10}(Re/Ra^{1/2}) = -1.432 - (3/4)\log_{10}Pr$ (solid line, hardly distinguishable from the dashed one); the fit value for the prefactor from linear regression is $-1.413 \pm 0.005$ [12].

4 Conclusion

We have seen a global theory for Ra and Pr dependence of Nu and Re. We should remember what is the main idea behind the theory and how do we take into account different sizes of boundary layers and their transitions into other regimes.

The main result of GL theory involves 6 dimensionless prefactors which are slightly de-
Figure 6: Nusselt number versus Rayleigh number. Reduced Nu for Γ = 1, obtained using water (Pr ≈ 4.4) and copper plates, as a function of Ra. Open symbols represent uncorrected data. Solid symbols, after correction for the finite plate conductivity. Circles [19], squares [20]. The upwards and downwards triangles are upper and lower bounds on the actual Nu at large Ra. The diamond originate from an estimate. The solid line is the Grossmann-Lohse prediction [3].

Dependent on the aspect ratio and are therefore not entirely universal. The results of the theory are very good, the prediction of the theory is in agreement with data obtained from simulations and experiments. We can conclude that this field which has been discussed for centuries will still continue to grow rapidly in the future.

References