Abstract

In this seminar we first take a look at the definition of fractal structures. Then we take a look at how they are divided into subcategories, and where do we find occurrences of each type of fractals in nature. And finally we see how fractal structures can be, and have been applied in science today.
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1 Introduction

At first the concept of fractals was used only in mathematics to describe self similar structures. Over time this concept found its way into other regions of science, because they could describe structures in nature that conventional mathematical tools could not. Once we understand why and how these fractal structures occur in nature, we can then use them in applications to create things otherwise impossible or hard to achieve. For instance with the aid of fractals and computers we can generate procedural terrain very close to real terrain, we can also make a very compact antenna that can receive multiple wavelengths, or simulate natural phenomena like lightning with accuracy we could not achieve without knowledge of fractal dimension; a property of fractals.

To understand fractals we must first know what defines them. What properties they share and how do we differentiate one fractal from another. We must learn to recognize them in mathematics and in nature. Finally we must know how to replicate them with mathematical or physical simulations.

2 Brief History

The first mathematic tools for the description of fractal structures were developed by Gottfried Leibnitz in the 17. Century, when he wrote about recursive self-similarity. He and other mathematicians after him only touched the surface of fractal structures, because there were no mathematical tools for describing and analyzing them. Some of the mathematicians of this time gave fractals the name "mathematical monsters". It took two centuries for the first published fractal to emerge. Karl Weierstrass 1872 published a function (equation 1) which graph was a fractal (figure 1).

\[ f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \] (1)

After him in 1883 Georg Cantor published a subset of the interval [0,1] which was a fractal and is today named the Cantor set. This type of fractal, which was described by an iteration of a set of rules, inspired a great breakthrough in 1904 by Helge von Koch. He described and constructed many new fractals with geometric recursive rules including his most known fractal the Koch snowflake. 10 years later Waclaw Sierpinski followed his example and constructed more fractals including the Sierpinski triangle. The next breakthrough was in 1918 when Pierre Fatou and Gaston Julia designed new fractals with a different approach. They used iterative functions on the complex plane to create the Julia set and others. At the same time Felix Hausdorff defined the fractal dimension which does not need to be an integer.

The main problem of these mathematicians was the representation of fractals. They had to be drawn by hand and many details were missed because of this. In 1960 Benoit B. Mandelbrot changed this with the use of computers. He collected the work of all mathematicians before him and joined it under the word he created: Fractal. The word was made from the Latin word fractus, which means broken or fractured. Because of his work he is often named the father of fractal geometry.
3 Properties of fractal structures

There is no unified theory that would describe all fractals. However there are common properties that all fractals should have. These are:

- Selfsimilarit y over all scales.
- Fine and detailed structure at arbitrary small scales.
- Irregularities, that cannot be described by euclidean geometry.
- Fractal dimension.

3.1 Self-similarity

Typical example of a self-similar structure in nature is the cauliflower (figure 3). If we zoom into a part of a cauliflower we see a structure that is very similar to the original. If we continue to zoom further into the cauliflower the self-similarity will eventually end. The same happens to all natural fractals, while mathematical fractals do not have this problem. This is why we divide self-similarity in three main categories:

1. Exact self-similarity.
Exact self-similarity is the strongest type of self-similarity encountered only in theoretical fractals. These fractals contain exact copies of themselves through all scales. Typical examples are Koch’s snowflake and Sierpinski’s gasket.

Approximate self-similarity is the less restrictive type of self-similarity. Fractals of this type contain distorted or degenerate copies of themselves. This type of self-similarity corresponds to most fractals found in nature and also some theoretical fractals, like the Mandelbrot set (figure 4).

![Figure 4: Mandelbrot set and a magnification of a primary bulb](image)

Statistical self-similarity is the weakest of the three. In these fractals only the numerical and statistical properties are preserved across all scales. Statistical self-similarity is mostly found in fractals that are generated by some sort of random behavior. One such fractal is the Brownian tree, which is generated by placing a stationary seed and then choosing a point and randomly propagate it around until it encounters the seed. It then becomes a part of the seed and we choose a new point to propagate. This can be done in 2, 3 or even higher dimensions and the principle remains the same. The Brownian tree is not just a computer generated structure, but it also has its analogues in nature. An example is the process of diffusion limited aggregation, which creates Brownian trees. The core principle of diffusion limited aggregation is propagation of particles via Brownian motion which, when they encounter the "seed", stick to it. Figure 5 shows the example of a computer generated Brownian tree and a Brownian tree generated by electrodeposition of copper sulfate.

![Figure 5: Computer generated Brownian tree, and a tree made by diffusion limited aggregation](image)

3.2 Fractal dimension

In his article “How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension” Mandelbrot discussed that the length of a coast strongly depends on the length of the measuring stick. The shorter the stick is the more details we can measure and thus the total length gets greater.

We encounter a similar problem with all Fractal structures. But some fractals are more detailed than others and their length grows quicker when reducing the "stick" size. To describe this behavior the fractal dimension was defined. The fractal dimension enables us to compare how similar or different two fractals actually are. A usage of this would be to compare the fractal dimension of a healthy heart beat
to an unhealthy one, or the fractal dimension of a simulated lightning to a real one to see how close are the simulations to reality.

The fractal dimension was introduced in 1918 by Felix Hausdorff and is also called the Hausdorff dimension, and is defined as follows. Let $X$ be a metric space. If $S \subset X$ and $d \in [0, \infty)$, then the $d$-dimensional Hausdorff content of $S$ is defined by

$$C^d_H(S) := \inf \left\{ \sum_i r_i^d : \text{there is a cover of } S \text{ by balls with radii } r_i > 0 \right\}. \tag{2}$$

And the Hausdorff dimension of $X$ is defined by

$$\dim_H(X) := \inf \{ d \geq 0 : C^d_H(X) = 0 \}. \tag{3}$$

We understand this as having the number of $N(r)$ balls with radius at most $r$ to cover $X$. If $r$ is big $N(r)$ is small but as $r$ gets smaller $N(r)$ grows (Figure 7). The Hausdorff dimension of $X$ is the number $d$ such that $N(r)$ grows as $\frac{1}{r^d}$.

We can find it difficult to do this on fractals with approximate or statistical self-similarity. We can then use a definition (or more accurately a method) that is worded differently but has the same results. Instead of covering the fractal with spheres, we divide the area into a grid of hypercubes of dimension $\epsilon$. Then we define $N(\epsilon)$ the number of hypercubes that contain part of the fractal. The fractal dimension of the fractal is than calculated as:

$$\dim = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \frac{1}{\epsilon}} \tag{4}$$
This is also called the "box counting dimension" and often used as a computer algorithm for determining the fractal dimension. Compared to the Hausdorff dimension the box counting method will give exactly the same results, except in some well known special cases.

One could ask if there is a difference if we cover the set with spheres or cubes or any other objects for that matter. The answer is no. While using spheres in the box counting method would be impossible because we cannot completely cover an area with spheres, we could use hexagons in two dimensions or tetrahedrons in three dimensions. The only thing that matters is that the object we are using has a correct dimension. Two for fractals in two dimensions and three in three dimensions and so on.

We can also see a small analytical example of how the method works on the Cantor set. The Cantor set is a set on the interval \([0, 1]\). It is generated by taking the line from 0 to 1, then dividing it into three pieces and removing the middle. We then repeat this on the two smaller lines. What remains is the Cantor set (Figure 8).

![Figure 8: The Cantor set \([8]\).](image)

We can easily calculate the fractal dimension of the Cantor set by setting \(\varepsilon = \frac{1}{3^k}\) where \(k\) is an integer. This makes \(N(\varepsilon)\) equal \(2^k\). The fractal dimension is then

\[
\dim = \frac{\log 2^k}{\log 3^k} = \frac{\log 2}{\log 3} = 0.6309
\]  

If we have a fractal with exact self-similarity the original definition can be greatly simplified. Let us start from a non-fractal example and build on that. Let us take a line and divide it into \(N\) equal pieces. Each of these pieces must be scaled \(N\) times to be the same as the original. Now we consider a square. We cannot divide a square into 2 equal smaller squares, but we can divide it into 4 or 9 or generally into \(N^2\). If we take one of these \(N^2\) squares we have to scale it \(N\) times in every dimension to get the original square. A cube can be divided into \(N^3\) cubes which must be scaled \(N\) times in every dimension to get the original. With this in mind we define the fractal dimension as

\[
\dim = \frac{\log (\text{self-similar parts})}{\log (\text{scale})}
\]  

This definition is derived directly from the Hausdorff definition. Using only the fact that the fractal is exactly self-similar.

The Cantor set is exactly self-similar so we can use this tool on it. When we look at the Cantor set we see that it has two parts (left and right). We must scale each of these parts 3 times to get the original set. The dimension is then \(\dim = \frac{\log 2}{\log 3}\)

But if we look closer the Cantor set actually has 4 parts. Each of which must be scaled 9 times to get the original set. This is no problem, because it does not change the result \(\frac{\log 4}{\log 9} = \frac{\log 2^2}{\log 3^2} = \frac{\log 2}{\log 3}\).

The Sierpinski triangle has a fractal dimension of \(\frac{\log 3}{\log 2}\) because each of its 3 smaller triangles must be enlarged 2 times to get the original.

4 Examples of fractal structures

For comparison and historic reasons it is important that we familiarize ourselves with the typical fractals. This will give us the ability to compare well known fractals to those that we are just discovering and
make are job easier.

4.1 Mathematical fractal structures

Mathematical fractal structures are the most important for comparison, because they give us the strongest tools for further analysis. To avoid creating a large list of countless mathematical fractals we will look at the different methods of making fractals and some example fractals of those methods. These groups are not strict definitions and do not cover every fractal.

The first group of fractals are those generated by iterating a set of replacement rules. As an example we look to the Sierpinski carpet. Sierpinski carpet is created by having a full square and dividing it into 9 equal, smaller squares, and removing the one in the middle, we then continue this in each of the 8 remaining squares. Figure 10 also shows the construction of the Koch snowflake. Other typical examples of this group are Sierpinski triangle, Menger sponge, cantor set, fractals generated by means of L-systems and space filling curves. L-system consists of an alphabet (symbols) and production rules that expand each symbol into some larger string of symbols.

Space filling curves are a group of their own. The main property of these fractal curves is that they cover the whole n-dimensional space they are in. They are also named Peano curves after their discoverer Giuseppe Peano (1858-1932). Examples for 2 dimensional space filling curves are the Peano curve (Figure 12), Hilbert curve, Gosper curve, Dragon curve. The Peano curve and the Hilbert curve as well as some others can be modified to work in three or even more dimensions.

The second large group of fractals is the escape-time fractals. These are constructed by having an initial set and a iterative function. We then choose a point and put it in the function, if the point after
infinite iterations escapes the set it is removed from the initial set. We do this for every point in the initial set and what remains of the initial set can be fractal, depending on the set and the function. Mandelbrot took the initial set of all \(c\) in the complex plane where \(|c| < 2\) an the iterative function \(z_{n+1} = z^n + c\) where \(z_0 = 0\) and \(c\) is a point from the set. The Julia set has the same initial set and function except that \(z_0\) is now a point from the initial set and \(b\) is just a parameter for that particular Julia set. Other examples are Newton fractal, burning ship fractal and nova fractal.

Brownian trees, Levy flight fractals, self-avoiding walk fractals, fractal landscapes are a part of a group of fractals that are generated by stochastic rules and exhibit only statistical self-similarity.

### 4.2 Fractal structures in nature

There are so many fractal structures in nature that it is to choose where to begin. If we begin with ourselves, many of our internal organs have a fractal structure. The biggest example can be the lungs. With their fractal dimension 2.97 they nearly completely fill the 3 dimensions. The reason for this is in the need for more exchange surface for the oxygen and carbon dioxide. The surface of the brain is folded in a fractal structure and has a dimension of approximately 2.76. Arteries also have a high fractal dimension of 2.7 so that their coverage is more uniform.

Ferns, trees and leaves are an example of a fractal structures. The reason for the fractal structures of leaves is the same as for arteries. But for ferns and trees the reason could perhaps be something different. They may have some iterative "rule" that grows them in such a way.

Fractals can also be found as purely physical phenomena. Such as lightning, the edge of Brownian motion, snowflakes and other crystals, fractures and so on. For some of them we can find an explanation, like crystals. When crystals form the forces at work always have the same symmetries on all of the parts of that crystal. For some the reason for their fractal nature is hard to explain.

Fractal structures in nature are not "perfect" fractals. If we zoom deep enough into their structures their detail and self similarity ends. For example the cauliflower seems fractal, but if examine it with a
magnifying glass, we can see that it is no longer fractal at that scale. The question is until what scale must a structure exhibit fractal nature that it can be called a fractal structure. This has no simple answer. In most cases, with fractals in nature, it is enough that the structure is fractal as far as we can see with the naked eye.

Figure 14: Fractal structures found in nature [14].

Figure 15: Crystal of fluorite exhibiting fractal properties [14].
5 Fractal quantum wave function of a particle

I describe this specific example in an attempt to further illustrate that fractal structures can in fact be found everywhere and do not have to be a consequence of complex influences.

Let us consider a particle in an infinite potential hole where the potential $V$ is infinite except in the interval $[0, 1]$ where it is zero. The Hamiltonian and eigenstates of such a particle in dimensionless form are

$$H = -\frac{\partial^2}{\partial x^2},$$

$$\psi_n(x, t) = \sqrt{2} \sin k_n x e^{-itE_n},$$

$$k_n = n\pi,$$

$$E_n = n^2 \pi^2,$$

$$k_n = \sqrt{E_n} \quad (7)$$

For simplicity let us say that the particle is in its base state. Now we instantly insert an infinite barrier at $0 < \epsilon < 1$, so that the new potential is written as

$$V(x) = \infty \delta(\epsilon - x) + \begin{cases} 
\infty & \text{if } x < 0 \text{ or } x > 1 \\
0 & \text{otherwise} \end{cases} \quad (9)$$

An instant insertion means that we did not affect the original wave function of the particle in any way. If we wish to propagate this picture in time we must expand the wave function with the use of the new eigenfunctions. Because of $\epsilon \to 1 - \epsilon$ symmetry we can now do all further math only for the $0, \epsilon$ part of the hole. Eigenfunctions for this part are $\mu_n(x) = \sqrt{\frac{2}{\epsilon}} \sin(k_n x / \epsilon)$. The series expansion is

$$\psi'(x) = \sum_{n=1}^{\infty} a_n \mu_n(x) \quad (10)$$

$$a_n = \int_0^\infty \psi^*(x) \mu_n(x) dx \quad (11)$$

$$a_n = -\frac{2(-1)^n n \sqrt{\epsilon} \sin(n\pi)}{n^2 \pi - \pi \epsilon^2} \quad (12)$$

This new function $\psi'(x)$ converges point to point to $\psi(x)$ on the interval $[0, \epsilon]$, which means that the functions only differ at the point $\epsilon$, where we get a discontinuity. Now we can propagate this in time. We immediately notice infinitely small detail (depending on how many terms in the expansion we actually do) forming on the wave function. On closer inspection we actually find that it became a fractal. This is not because we can only take so many terms in the expansion, in fact taking more terms only further the detail of the function. Some explain this as introducing an infinite amount of energy into one point when we instantly insert the barrier, which then propagates over the whole area and makes the function fractal.

This phenomena has been observed by Michael Berry who discovered the any discontinuity in the wave function will cause it to have a fractal waveform when we propagate it in time.
Applications of fractal structures

A lot of applications can be found in computer science. Like terrain generation, music generation, data compression, simulations... Data compression assumes that the data has a lot of similar parts and uses that to write only one of those parts to memory and transformation rules for the other parts. This is very effective in pictures of nature for example.

Sometimes the fractal dimension of a phenomenon gives us that one extra parameter to compare. A study was done that shows that measuring the fractal dimensions of a heart can give better reliability in determining healthiness. Lightning strikes can be more accurately simulated if we take into account the fractal dimension of lightning and try to replicate it in the simulation.

A widely used application is the fractal antenna (Figure 17). The fractal antenna instead of being cut to specific length for a specific frequency can operate on multiple frequencies at once or even over a wide spectrum of frequencies. Their small size and accessibility paved their way into many applications like cell phones and in cars where one very small fractal antenna can serve several telecommunication services. The figure 18 shows a comparison of SWD and gain of a normal monopole antenna and a monopole antenna coated with a fractal metamaterial [16]. This shows that simply coating the antenna with this fractal metamaterial gives a roughly 3 times larger bandwidth and about 3dB increase in gain across the board.

Another application are fluid and gas injectors (figure 19). The aim of this injectors is to inject fluid or gas gently and uniformly. The tree design of such injectors enables them to be easily scaled for larger applications and have no moving parts for easier construction.

There are many other applications of fractal structures, and many are yet to come, because the concept of using fractals in applications is relatively new. There unique properties give them an edge where a large circumference or surface area is required in a limited space, and their strong connection with chaotic systems will continue to give us insight in that area.
Figure 17: Fractal antenna [15].

Figure 18: Fractal metamaterial and comparison between monopole antenna and an antenna coated with this metamaterial [16].

Figure 19: Fractal injector [15].
7 Conclusions

Even though fractal structures and fractal geometry are relatively new concepts, they have been used in many different fields of science. Even some new fields have been made that use fractal as their primary means of analysis, such as fractal electrodynamics, fractography, fractal in soil mechanics and so on.

In the future, as these new fields develop and progress, they will hopefully help our understanding of nature overall and bring new applications of fractal structures, that will be as widely used the fractal antenna is today.
References

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