Seminar

Chaos in dynamical systems

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Abstract

We make a quick introduction to chaos in dynamical systems. We present some basic concepts for understanding and describing dynamical systems that are chaotic. As an example we first consider the logistic map, which is a mathematical prescription with a clear physical background, and has a chaotic dynamics. Then we consider two more physical systems: the population model of a laser and scattering of particles by a potential. The dynamics of level population in a laser is chaotic if we add some periodical disturbance. We choose the simplest one: kicking the system with Dirac delta functions. It turns out that the scattering on a chosen potential becomes chaotic if the energy of the particle is sufficiently small. In this case, chaos can be seen in a sensitive dependence of the scattering angle on the impact parameter.

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1 Introduction

1.1 Dynamical systems

A dynamical system may be defined as a deterministic mathematical prescription for evolving the state of a system forward in time [1]. Time here may either be a continuous variable, or it may be a discrete integer-valued variable. An example of a dynamical system in which time is continuous is a system of $N$ first-order ordinary differential equations,

$$\frac{dx(t)}{dt} = F[x(t)],$$

(1)

where $x$ is an $N$-dimensional vector and $t$ is time. The space $(x^{(1)}, x^{(2)}, x^{(3)}, ...)$ is referred to as phase space, and the path in phase space followed by the system as it evolves with time is referred to as an orbit or trajectory. Also, it is common to refer to a continuous time dynamical system as a flow. In the case of discrete integer-valued time an example of dynamical system is a map which can be written in vector form as

$$x_{n+1} = M(x_n),$$

(2)

where $x_n$ is $N$-dimensional, $x_n = (x_n^{(1)}, x_n^{(2)}, ..., x_n^{(N)})$. Given the initial condition $x_0$, we generate the orbit of the discrete time system: $x_0, x_1, x_2, ...$

1.2 Chaos

For a motivation we consider two experiments, which examine the chaotic behavior of physical systems. As a first example, we consider the experiment of Shaw (1984) [2]. In this experiment, a slow steady inflow of water to a faucet is maintained. Water drops fall from the faucet, and the times at which successive drops pass a sensing device are recorded (Figure 1). Thus, the data consists of the discrete set of times $t_1, t_2, t_3, ..., t_n$ at which drops were observed by a sensor. From these data, the time intervals between successive drops can be formed, $\Delta t_n = t_{n+1} - t_n$. When the inflow rate to the faucet is sufficiently small, the time intervals $\Delta t_n$ are all equal [1]. As the inflow rate is increased, the time interval sequence becomes periodic with a short interval $\Delta t_a$ followed by a longer interval $\Delta t_b$, so that the sequence of time intervals is of the form $..., \Delta t_a, \Delta t_b, \Delta t_a, \Delta t_b, \Delta t_a,...$. We call this a period two sequence since $\Delta t_n = \Delta t_{n+2}$. As the inflow rate is increased further, periodic sequences of longer and longer periods were observed, until, at sufficiently large inflow rate, the sequence $\Delta t_1, \Delta t_2, \Delta t_3,...$ apparently has no regularity. This irregular sequence is argued to be due to chaotic dynamics. Several different expressions are used for this phenomenon of period doublings, but we will refer to it by period doubling cascade or simply bifurcations.

![Figure 1: Experiment of Shaw (1984). A slow steady inflow of water to a faucet is maintained. Water drops fall from the faucet, and the times at which successive drops pass a sensing device are recorded. When the inflow rate to the faucet is sufficiently small, the time intervals between recorded drops are all equal. As the inflow rate is increased, period doubling can be observed [1].](image)
As a second example, we consider the Rayleigh-Bénard experiment in liquid helium. This experiment was first done by Albert Libchaber in 1978 [3]. A fluid (in this case liquid helium) is contained between two rigid plates and subjected to gravity. The bottom plate is maintained at higher temperature \( T_0 + \Delta T \) than the temperature \( T_0 \) of the top plate. As a result, the fluid near the warmer lower plate expands and creates a tendency for this fluid to rise. Similarly, the cooler more dense fluid near the top plate has a tendency to fall. One can therefore observe a steady convective circular motion of a single frequency of a fluid [1] as shown in Figure 2a. If \( \Delta T \) is increased, the period doubling cascade can be observed [4], similar to that in the experiment of Shaw. Apart from period doubling, Libchaber also showed that the amplitudes of individual modes can be predicted. If a temperature difference is increased further, the chaotic turbulent flow arises. This kind of motion has no periodicity, therefore the spectrum is nothing but noise. Saying that, observing the transition to chaotic regime can be done simply by using spectral analysis. One can see that with period doublings the spectrum contains more and more peaks (Figure 2b), which leads to noise when the dynamics becomes non-periodical.

Figure 2: Rayleigh-Bénard experiment in liquid helium (a). A fluid is contained between two plates. At very small \( \Delta T \) the circular motion of a single frequency can be observed. When \( \Delta T \) is increased new frequencies arise, which affect the spectrum in terms of new peaks (b) [4].

### 1.3 Attractors and sensitivity on initial conditions

In Hamiltonian systems such as arise in Newton’s equations for the frictionless motion of particles, there are choices of the phase space variables such that phase space volumes are preserved under the time evolution. We call such a volume-preserving system conservative. On the other hand, if the flow does not preserve the volume, and cannot be made to do so by a change of variables, then we say that the system is nonconservative. It is an important concept in dynamics that dissipative systems typically are characterized by the presence of attracting sets or attractors in the phase space. These are bounded subsets to which regions of initial conditions of nonzero phase space volume asymptote as time increases [1]. Conservative dynamical systems do not have attractors.

A defining attribute of an attractor on which the dynamics is chaotic is that it displays exponentially sensitive dependence on initial conditions [1]. Consider two nearby initial conditions \( x(0) \) and \( x(0) + \epsilon(0) \), and imagine that they are evolved forward in time by a continuous time dynamical system yielding orbits \( x_1(t) \) and \( x_2(t) \) (Figure 3). At time \( t \) the separation between the two orbits is \( \epsilon(t) = x_2(t) - x_1(t) \). If, in the limit \( |\epsilon(0)| \to 0 \), at large \( t \), the orbits remain bounded\(^1\) and the difference between the solutions \( |\epsilon| \) grows exponentially, then we say that the system displays a sensitive dependence on the initial conditions and is chaotic [1].

The measure for the dependence on initial conditions is known as the Lyapunov exponent. In one dimension it is defined as [5]

\[
\lambda = \lim_{t \to \infty} \lim_{\epsilon(0) \to 0} \frac{1}{t} \log \frac{|x_1(t, x_1(0)) - x_2(t, x_1(0) + \epsilon(0))|}{|\epsilon(0)|}.
\]  

(3)

Let us now examine three representative examples of chaotic dynamical systems. The first is more mathematical, while the other two have more physical background.

\(^1\)By bounded solutions, we mean that there is some ball in phase space, \(|x| < R < \infty\), which the solutions never leave. The reason for the restriction is that, if orbits go to infinity, it is relatively simple for their distances to diverge exponentially.
Figure 3: Two nearby initial conditions evolve in time by a continuous dynamical system. If, in the limit 
\( |\Delta(0)| \to 0 \), at large \( t \), the orbits remain bounded and the difference between the solutions \( |\Delta| \) grows exponentially, then we say that the system displays a sensitive dependence on initial conditions and is chaotic [1].

2 The logistic map

We now consider an idealized ecological model for the yearly variations in the population of an insect species [1]. In this case of non-Hamiltonian system it is not valid to say weather this system is conservative or dissipative. Imagine that every spring these insects hatch out of eggs laid the previous fall. They eat, grow, mature, mate, lay eggs, and then die. Assuming constant conditions each year, the population at year \( n \) uniquely determines the population at year \( n+1 \). We can imagine that for each insect, on average, there will be \( r \) eggs laid, each of which hatches at year \( n+1 \). This yields a population at year \( n+1 \) of \( z_{n+1} = rz_n \). This also means an exponentially increasing population, \( z_n = r^n z_0 \) if \( r > 1 \). However, if the population is too large, the insects may begin to exhaust their food supply as they eat and grow. Thus some insects may die before they reach maturity. Hence the average number of eggs laid per hatched insect will become less than \( r \) as \( z_n \) is increased. The simplest possible assumption considering this overcrowding effect would be to say that the number of eggs laid per insect decreases linearly with the insect population, \( r[1 - (z_n/\bar{z})] \), where \( \bar{z} \) is the insect population at which the insects exhaust all their food supply such that none of them reach maturity and lay eggs. This physical picture can be expressed by a one-dimensional map

\[
x_{n+1} = rx_n(1 - x_n),
\]

where \( x = z/\bar{z} \). The map function \( M \) as defined by equation (2) is thus \( M(x) = rx(1-x) \). The maximum of \( M(x) \) occurs at \( x = 1/2 \) and is \( M(1/2) = r/4 \). Thus, for \( 0 \leq r \leq 4 \), if \( x_0 \) is in \( [0,1] \), then so is \( x_{n+1} \), and the orbit remains in \( [0,1] \) for all subsequent time. We restrict our considerations to \( 1 \leq r \leq 4 \) and \( x \in [0,1] \).

For most parameters \( r \) the sequence (4) converges to one value, which depends on \( r \) (Figure 4b), but is the same for all starting points (Figure 4a). We call this period one orbit.

On the other hand, we notice that for some parameters \( r \) the sequence (4) does not converge in the limit \( n \to \infty \), but jumps between 2, 4, 8, 16, ... points instead (Figure 5a). We call this period–two, –four, –eight, –sixteen,... orbit. At some \( r_\infty \) the orbit is no longer periodical but chaotic (Figure 5b).

The period–one orbit is stable for \( 1 < r < r_0 = 3 \), while the period–two orbit is stable in the range \( r_0 < r \leq r_1 \). As \( r \) increases through \( r_1 \), the period–two orbit doubles to a period–four orbit. This process of period doublings continues, successively producing an infinite cascade of period doublings within range \( r_{m-1} < r \leq r_m \), in which period–2^m orbits are stable. The length in \( r \) of the range of stability for an orbit of period 2^m decreases approximately geometrically with \( m \). Generally

\[
\delta \equiv \frac{r_m - r_{m-1}}{r_{m+1} - r_m} \xrightarrow{m \to \infty} 4.669201...
\]

where \( \delta \) is a universal constant known as the Feigenbaum constant.
Figure 4: For most parameters $r$ the sequence (4) converges to one value (a). Convergence point is independent on $r$ (b), but is the same for all starting points. We call this period–one orbit.

Figure 5: For some parameters $r$ the sequence (4) does not converge in the limit $n \rightarrow \infty$, but jumps between $2, 4, 8, 16, \ldots$ points instead (a). We call this period two, four, eight, sixteen,\ldots orbit. If we plot the convergence points relative to the parameter $r$ we get the so–called bifurcation diagram (b).

As we discussed in the previous section, the Lyapunov exponent is a good measure for dependence on the initial conditions. We notice that for the regions of parameter $r$ for which the system is non–chaotic, the Lyapunov exponent is practically zero while for parameters between those regions, the Lyapunov exponent is finite (Figure 6).

### 3 Population model of a laser

As an example of a nonconservative dynamical system we consider a kicked laser model. First we derive the equations, describing the population model of a laser with constant pumping (without kicking). The time evolution of the number of photons $f$ and excited atoms $a$ is given by [6]

$$
\frac{df}{dt} = Aaf - pf,
$$
Figure 6: The Lyapunov exponent as a function of parameter $r$. Where $\lambda = 0$ the system is non-chaotic and where $\lambda > 0$, the system is chaotic [5].

\[
\frac{da}{dt} = -Aaf - \lambda a + R,
\]

where $A$ is the Einstein coefficient of stimulated emission, $\lambda$ is the probability for an atom to decay coherently, $p$ is the loss coefficient and $R$ is the power of constant pumping. Here we are only interested in dynamical system itself given by previous two equations so we will not go into details about physical background of a laser. Further physical details can be found in [6]. The system of equations (6) and (7) can be written in non-dimensional form as

\[
\frac{d\phi}{d\tau} = \frac{1}{S}(\alpha\phi - \phi),
\]

\[
\frac{d\alpha}{d\tau} = -S(\alpha\phi - 1) + Z(\alpha\phi - \alpha),
\]

where $S = \sqrt{AR}/p$ and $Z = \lambda/\sqrt{AR}$. The time $\tau$ is measured in units of $\sqrt{AR}$, the number of photons $\phi$ in units of $(AR - \lambda)p/Ap$ and the number of atoms $\alpha$ in units of $p/A$. We model the kicked laser by adding the additional expression in equation (9) which describes the kicking of a laser and is simply a periodic repetition of Dirac delta functions (Dirac comb). The final system of differential equations is therefore

\[
\frac{d\phi}{d\tau} = \frac{1}{S}(\alpha\phi - \phi),
\]

\[
\frac{d\alpha}{d\tau} = -S(\alpha\phi - 1) + Z(\alpha\phi - \alpha) + C \sum_{i=-\infty}^{\infty} \delta(\tau - iT),
\]

where $C$ is a non-dimensional amplitude of a kick and $T$ is a non-dimensional period of kicking. We solve the system by using the RK4 method with adaptive step size. With some parameters the motion of the system in its phase space is periodical (Figure 7a), while with others the motion is non-periodical (chaotic) (Figure 7b). In case of periodical motion, the trajectory in the limit $t \to \infty$ is called an attractor and is a limit cycle to which trajectories with different initial conditions converge. On the other hand, in the case of non-periodical motion, the trajectory in the limit $t \to \infty$ never reaches the attractor but reaches almost every state in some region of phase space.

The system has four independent parameters and is therefore very complicated. So it would be impractical to scan the whole space of parameters. One can see that two parameters ($S$ and $Z$) belong to the laser itself, while the other two ($C$ and $T$) come from kicking. While the system without kicking is not chaotic, one can suspect that it is the parameters $C$ and $T$ that cause the transition of the system.
Figure 7: With some parameters the motion of the system in its phase space is periodical (a). In this case the trajectories with different initial conditions converge to the limit cycle called the attractor. With some other parameters the motion is non–periodical (chaotic) (b). In this case the trajectory in the limit \( t \to \infty \) never reaches the attractor but reaches almost every state in some region of phase space.

to chaos. We therefore keep \( S \) and \( Z \) constant \((S = 1/12, Z = 0)\) and scan only a part of the subspace of parameters \((1 < T < 50 \text{ and } 0 < C < 1)\).

When it comes to systems with periodical disturbance we introduce a new concept of observing the system, called the Poincare map. It is an inersection of a periodic orbit in the phase space of a continuous dynamical system with a certain lower dimensional subspace, called the Poincare section, transversal to the flow of the system \([7]\).

In the case of the kicked laser, rather than observing the system continuously in some time interval, we observe the state of a system \((\alpha, \phi)\) at discrete times \( t_n \), so that \( t_{n+1} - t_n = T \). We choose to observe a system (number of excited atoms \( \alpha \)) just after every delta kick. What we get is a sequence \( \alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots \) \( n \gg 1; \quad \alpha_n \equiv \alpha(nT + 0) \),

\[(12)\]

where \( n \gg 1 \) means that we start observing the system after many delta kicks, when the trajectory in the phase space approaches the limit cycle (attractor) if the motion is periodic. What happens is that at most values of parameter \( C \) the sequence (12) has period one and therefore is \( \alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots \). At some critical \( C_1 \) the sequence becomes \( \alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+3}, \ldots \) which is a period–two orbit. Then at some critical \( C_2 \) the period–four orbit occurs \((\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+3}, \alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+3}, \ldots)\), etc. This is the same period doubling phenomenon that we previously encountered in the experiment of Shaw, the Rayleigh-Benard experiment in liquid helium and in the logistic map. The Feigenbaum scenario for the route to chaos is obviously very common in physical systems.

The number of periods for the chosen set of parameters \( T \) and \( C \) is shown in Figure 8. The red sections denote the parameters for which the number of periods is infinite and the motion of the system is chaotic.

Now that we know for which parameters \( C \) and \( T \) the motion of the system is chaotic, we can look what happens if we change \( Z \) and \( S \). From the definitions of these two parameters we see that the role of \( Z \) in equations (10) and (11) is qualitatively the same as the role of \( 1/S \). They both represent some kind of damping. We therefore leave \( S \) constant and only check the influence of \( Z \) on the transition of a system to chaos.

We notice (Figure 9) that with higher \( Z \) (and constant \( S = 1/12 \) and \( T = 25 \)) the bifurcation diagrams become poorer and the region of chaotic behavior is becoming smaller. At some critical \( Z \approx 0.035 \) (Figure 9c) the chaotic behavior of the system disappears completely for chosen parameters but bifurcations may still remain. But at higher \( Z \) the bifurcations disappear as well (Figure 9d). Damping obviously decreases the regions of chaotic behavior of a system. Highly damped systems therefore can not be chaotic.
As was mentioned before, the spectrum analysis can be used to discover whether the dynamics is chaotic or not. Figure 10 shows the power spectrum of the signal obtained with Poincare map (12). The set of parameters at which the system is non-chaotic were chosen ($0.562 < C < 0.57$ - green curve) and the set of parameters at which the system is chaotic were chosen ($0.688 < C < 0.692$ - the red curve). As was expected the spectrum of non-chaotic dynamics has some well defined peaks which correspond to periodicity of a sequence (12). On the other hand, the spectrum of chaotic dynamics has no distinct peaks and is therefore noisy.

Finally we choose some set of parameters $S$, $Z$ and $T$ for which the motion of the system is chaotic and determine the Feigenbaum constant (5) to prove that it is really universal. We approximately estimate it from the bifurcation diagram (Figure 9a). We obtain

$$\delta \approx \frac{C_4 - C_3}{C_5 - C_4} = 4.47,$$

which is just about 4\% from the exact value (4.669201...).
Figure 9: With higher $Z$ bifurcation diagrams get poorer and the region of chaotic behavior is getting smaller. At some critical $Z \approx 0.035$ (c) the chaotic behavior of the system disappears completely for chosen parameters but bifurcations may still remain. But at higher $Z$ bifurcations disappear as well (d).

4 Chaotic scattering

We consider the classical scattering problem for a conservative dynamical system. We deal with the frictionless motion of a point particle in a potential $V(x)$ which is zero or very small outside of some finite region of space which we call the scattering region. A particle moves toward the scattering region from a great distance, interacts with the scatterer, and then leaves the scattering region. If sufficiently far outside scattering region, the particle moves along a straight line (or an approximately straight line). The question to be addressed is how does the motion far from the scatterer after scattering depend on the motion far from the scatterer before scattering? As an example, let us discuss the scattering in two dimensions. The incident particle has a velocity parallel to the $x$-axis and a vertical displacement $y = b$ (the impact parameter). After interacting with the scatterer, the particle moves off to infinity with its velocity vector making an angle $\phi$ to the $x$-axis (the scattering angle). We wish to investigate the character of the functional dependence of $\phi$ on $b$.

We consider the potential [8]

$$V(x, y) = x^2 y^2 \exp[-(x^2 + y^2)]$$

shown in Figure 11a. This potential consists of four potential hills with equal maxima at $(1, 1), (1, -1), (-1, 1)$ and $(-1, -1)$. The maximum value of the potential is $E_m = 1/e^2$. For large distances $r =$
Figure 10: The spectral analysis of laser dynamics. If the motion of a system is periodic, one can observe well defined peaks in the spectrum (green curve). On the other hand, if the motion of a system is chaotic, the spectrum is noisy.

$$\sqrt{x^2+y^2} \text{ from the origin, } V(x, y) \text{ approaches zero rapidly with increasing } r. \text{ We write down the equation of motion for a particle of mass } m$$

$$\frac{d^2 r}{dt^2} = F,$$

where $F$ is the external force due to the potential $V(x, y)$ and can be calculated as $F = -\nabla V$. Written in nondimensional form the system of two second–order differential equations is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 2xy \exp \left[ - (x^2 + y^2) \right] \begin{bmatrix} y(x^2 - 1) \\ x(y^2 - 1) \end{bmatrix},$$

which can be rewritten as the system of four first–order differential equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} u \\ 2xy^2 \exp \left[ - (x^2 + y^2) \right] (x^2 - 1) \\ 2x^2y \exp \left[ - (x^2 + y^2) \right] (y^2 - 1) \end{bmatrix}.$$ (17)

For a chosen $b$ we obtain the corresponding scattering angle $\phi$ by integrating (17) by using the RK4 method with adaptive step size2.

The equations of motion are four-dimensional, but because the particle energy,

$$E = \frac{1}{2}mv^2 + V(x)$$

is conserved, we may regard the phase space as being three-dimensional. For example, we can regard the phase space as consisting of the three variables $x, y, \phi$, where $\phi$ is the angle the vector $v$ makes with the positive $x$-axis. These three variables uniquely determine the system state, because (18) gives $|v|$ in terms of $x$ and $y$ [1].

Figure 11b shows three orbit trajectories whose $b$ differ by $\sim 10^{-3}$. Orbits are very close together for the first two bounces but then they separate and have completely different scattering angles, one yielding scattering forward and the other two scattering downward. This is a lovely example of the sensitive dependence on initial conditions which implies that the dynamics is chaotic, as discussed in section 1.3.

2In problems with scattering it is useful to use an integrator with adaptive step size to save some time in very straight regions of trajectory.
Figure 11: (a) The potential (14) consists of four potential hills with equal maxima at $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ (a). The maximum value of the potential is $E_m = 1/e^2$. For large distances from the origin, $V(x,y)$ approaches zero rapidly with increasing $r$ (a). Orbit trajectories with very nearby initial conditions are very close together at first, but then they separate and leave the scattering region at completely different angles (b).

Figure 12: The scattering function for a case where $E < E_m$. In some regions of parameter $b$, the scattering function is a smooth curve, but in some regions the behavior is chaotic.

In the case where the incident particle energy $E$ is larger than $E_m$, the scattering function is a smooth curve. Furthermore, it is also found to be a smooth curve for all $E > E_m$ [1]. Figure 12 shows the scattering function for a case where $E < E_m$. We observe that the numerically computed dependence of $\phi$ on $b$ is poorly resolved in some regions. It might be taken to imply that the curve of $\phi$ versus $b$ varies too rapidly to be resolved on the scale determined by the spacing of $b$-values used to construct the figure. In this view one might still hope that sufficient resolution would reveal a smooth curve. That this is not the case can be seen in Figure 13 which shows the magnification of the unresolved region $-0.396 < b < -0.374$. Evidently the magnification of a portion of the unresolved region of Figure 13 by
a factor of order $10^2$ does not reveal a smooth curve.

So obviously in the regions where the curve is smooth, the system is non-chaotic by the basic definition of chaotic behavior described in section 1.3. On the other hand, in regions where the smooth curve is not revealed the behavior is chaotic.

Such scattering is simply a mechanical process, where we can simply imagine that we have a ball which bounces on four hills, but we can speculate that such scattering can be observed in experiments, where they observe the scattering on the nucleus. One possibility is also a scattering of an electron in crystals, but in that case we would have to take quantum effects into account.

## 5 Conclusions

We presented some basic concepts of chaotic motion of a dynamical system. The Lyapunov exponent was presented and understood as a quantity that measures the sensitivity on initial conditions, which is the fundamental property of chaotic dynamical systems. We showed that the spectral analysis also discovers whether the dynamics is periodical or chaotic. As an illustration of chaos in dynamical systems we studied three dynamical systems. With logistic map the bifurcations as a route to chaos were presented, that can also be observed in more physical system as it is the case with the population model of a laser with delta pumping. From the bifurcation diagram for the laser we calculated the Feigenbaum constant and showed that it is universal. In the last example - the chaotic scattering, we showed that the scattering angle as a function of the impact parameter is not necessarily a smooth curve and that it can be very sensitive on changes of the impact parameter if the energy of the scattering particle is smaller than the height of a given potential.
References


