Abstract

The magnetohydrodynamic (MHD) model of plasma provides a macroscopic description, where the plasma is considered as a single, conducting fluid. It can be derived from the kinetic theory, with the assumption, that we are only interested in the low-frequency, long-wavelength, magnetic behavior of plasma. Even though MHD is the least accurate macroscopic model, it is nevertheless still able to properly explain some of the macroscopic equilibrium and stability features of plasmas. One of the most important use-cases of MHD is the application to fusion plasmas, where the model can be used to accurately describe a state of macroscopic equilibrium.
1 Introduction

The research and description of various plasma phenomena has gained on its importance over the past decades, mainly due to the fact that the studies of various plasma confinement methods and instabilities are of essential importance for a successful operation of future thermonuclear reactors. One of the most fundamental challenges of this field of research is the study of various magnetic configurations, that should make it possible, to keep the plasma in a state of macroscopic equilibrium at very high temperatures. For such purposes we need a theory that gives a good description of plasma on the macroscopic scale in terms of measurable quantities such as, for example the electric current, temperature and pressure. One of the theories that fits these needs is the so called magnetohydrodynamic (MHD) theory of plasma. The MHD theory considers the plasma as a single, conducting fluid and is appropriate for the description of macroscopic phenomena that occur on rather slow timescales.

Thermonuclear reactors are machines used to confine and heat a hydrogen plasma, to temperatures high enough, to achieve the required conditions for a sustained fusion reaction. The reaction which offers best energetics for controlled thermonuclear\textsuperscript{1} fusion is \[ \text{D} + \text{T} \rightarrow \text{He} + n. \] \text{(1)}

The total energy output\textsuperscript{2} (17.5 MeV) is distributed between the alpha particle (3.5 MeV) and the neutron (14 MeV). The alpha particle is confined by the magnetic field and used to heat the fuel and sustain the reaction.

\textsuperscript{1}The nuclei are part of a plasma near thermal equilibrium.

\textsuperscript{2}The reaction is exothermic because the binding energy per nucleon generally increases with the mass number \( A \) for \( A < 50 \).
The MHD theory has in fact proven itself to be the one most useful in the description of basic equilibrium and stability properties of fusion plasmas, even though MHD actually gives the least detailed macroscopic description of plasma. In this paper we focus mainly on the MHD description of plasma and its application to plasmas inside thermonuclear reactors. Description of the plasma on the microscopic scale would involve the study of charged particle motions in electromagnetic field, that varies both spatially and in time. For the description of a various particle trajectories in a plasma the reader should refer to [2].

2 Basic plasma characteristics

Plasma is an ionized state of matter in which the portion of charged particles is large enough for the substance to be a good electrical conductor [3]. The charged particles (electrons) move in a sea of opposite charged particles (ions), which make the whole substance neutral of charge. The high electrical conductivity implies currents can flow in a plasma. These currents can interact with magnetic fields to produce the forces, that are needed for confinement. Another important property of plasmas is the existence of strong collective effects. These phenomena arise due to the long range of electromagnetic forces, that govern the dynamics of such an ionized state of matter. It should be emphasized, that the electrons play a much more important role in the occurrence of plasma collective effects, because they are much lighter than ions and are therefore able to respond more quickly to variations of the electromagnetic field. Plasma is sometimes referred to also as the fourth state of matter in addition to solids, liquids and gases.

Some of the basic plasma parameters are:

- the particle density \( n \)
- the temperature \( T \) of each species
- the collision frequencies \( \nu \) of each species
- the plasma frequency \( \omega_p \)

![Figure 1: Various plasma domains characterized by the electron temperature \( T \) (measured in electronvolts) and the particle density \( n \) [4].](image)

2.1 Charge neutrality

One of the fundamental properties of a plasma is its ability to maintain a state of charge neutrality. This is due to the fact, that the field caused by any charge imbalance is shielded, so that its influence is effectively restricted to within a finite range. If we slowly insert an additional charged test particle inside the plasma and calculate the electrostatic potential \( \phi \) from Poisson’s equation we get [5]

\[
\phi(r) = \phi_0(r)e^{-r/\lambda_D},
\]
where $\phi_0$ is the potential of the charge in empty space and $\lambda_D$ is the effective Debye length. It is defined with the species Debye lengths $\lambda_{\alpha}$

$$\frac{1}{\lambda_D^2} = \sum_{\alpha} \frac{1}{\lambda_{\alpha}^2}.$$  (3)

The Debye length of the species $\alpha$ is given by

$$\lambda_{\alpha} = \sqrt{\frac{\epsilon_0 k_B T_{\alpha}}{n_{\alpha 0} e_{\alpha}^2}},$$  (4)

where $e_{\alpha}$, $T_{\alpha}$ and $n_{\alpha 0}$ are the charge, temperature and equilibrium number density, respectively. $k_B$ is the Boltzmann constant and $\epsilon_0$ is the electric constant. The Debye length may take very different values, accordingly to the domain of plasma we are interested in. For fusion plasmas the typical lengths are of the order of $\lambda_D \sim 10 \mu m$. The summation in (3) is over all species that participate in the shielding. Since ions cannot move fast enough to keep up with an electron test charge, the shielding of electrons is only by other electrons, whereas the shielding of ions is by both ions and electrons. For $r \gg \lambda_D$ the test charge is completely screened by its surrounding shielding cloud.

The above analysis only makes sense if

$$n_{e 0} \lambda_D^3 \gg 1,$$  (5)

which means that there is a large number of electrons present inside the shielding cloud. The condition (5) represents one possible criteria for the definition of plasma.

### 2.2 Plasma oscillations

The strong electrostatic fields caused by charge imbalance cause oscillations about the equilibrium position. Let us assume that the ions remain more or less at rest throughout the perturbation and that the electrons are accelerated only by the electrostatic field $E$. We shall also take into account that the perturbation is small, so that we may write the electron number density as

$$n_e(r, t) = n_0 + n'_e(r, t),$$  (6)

where $n_0$ is the number density in equilibrium and $|n'_e| << n_0$. The above considerations lead us to

$$\frac{\partial^2 n'_e}{\partial t^2} + \omega_p^2 n'_e = 0,$$  (7)

where the natural frequency of oscillation $\omega_p$ is known as the plasma frequency and is defined as

$$\omega_p = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}.$$  (8)

Equation (7) shows that $n'_e$ varies harmonically in time at the electron plasma frequency. Any external fields with frequencies much smaller than $\omega_p$ are prevented from penetrating the plasma by the more rapid electron response, which neutralizes the field.

Collisions between electrons and neutral particles tend to damp the plasma oscillations and decrease their amplitude. If the electron oscillations are to play an important role, their collision frequency with the neutrals $\omega_{en}$ should be smaller than the electron plasma frequency [6]

$$\omega_{en} < \omega_p.$$  (9)

This is an additional criterion for the definition of plasma. If (9) is not satisfied, then the medium can be treated as a neutral gas.

---

3This is in most cases a good approximation, since the ions have a much larger mass and are unable to follow the fast electron oscillations.
2.3 Electrical resistivity

As shown in ([5], Chapter 13), the electrical resistivity of a plasma is roughly proportional to $T_e^{-3/2}$ and independent of the particle number density. This fact has some important consequences on the methods used to heat the plasma in thermonuclear reactors. The heating of plasma by driving the electric current inside the substance is effective at lower temperatures, but becomes less useful at higher temperatures, so that alternative approaches have to be used. These are:

- Neutral beam injection (injection of uncharged high-energy particles into the system)
- Electron/Ion cyclotron resonance heating (electromagnetic radiation with frequency equal to the resonant frequency of electrons/ions).

3 Kinetic description of plasma

3.1 Distribution function and the Boltzmann equation

Every plasma consists of a very large number of electrons and ions, therefore one can hardly observe the individual behavior of each particle. Fortunately, such detailed information is redundant, if we are only interested in the macroscopic properties of a plasma. Instead of keeping track of each particle, it is adequate to develop a less detailed statistical description. This approach characterizes classes of particles having the same position $r$ and velocity $v$ in phase space, but doesn’t keep track of individual trajectories.

The function describing the density of particles having the same $r$ and $v$ at some time instant is the *single-particle distribution function* and is denoted by $f(r, v, t)$. It is defined as [6]

$$f_\alpha(r, v, t) = \frac{d^6 N_\alpha(r, v, t)}{d^3r d^3v}.$$  \hspace{1cm} (10)

$d^3r d^3v$ represents a small 6-dimensional box in phase space around the point $(r, v)$, $d^6 N_\alpha(r, v, t)$ corresponds to the number of particles inside this box and index $\alpha$ denotes the particle species.

The single-particle distribution function is a reduction of the many-particle distribution function, obtained by integrating over $N - 1$ particles of our statistical ensemble.

By the particle species we usually mean either the electrons or ions, although a plasma may consist of more than just one ion species. This is the case, for example in a D-T fusion plasma.
The normalization of $f_\alpha(r,v,t)$ is the number density $n_\alpha(r,t)$, which is the integral of the distribution function over velocity space.

The equation of motion for $f_\alpha(r,v,t)$, i.e. its total time derivative, is given by the *Boltzmann equation* [5]

$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + v \cdot \frac{\partial f_\alpha}{\partial r} + \frac{1}{m_\alpha} F \cdot \frac{\partial f_\alpha}{\partial v} = \sum_\sigma C_{\alpha\sigma}(f_\alpha). \quad (12)$$

The operator $C_{\alpha\sigma}$ is the collision operator $^5$ and represents the rate of change of $f_\alpha$ due to collisions between species $\alpha$ and $\sigma$. The force $F$ is in our case given by the Lorentz force

$$F = e(E + v \times B). \quad (13)$$

The fields $E$ and $B$ can be decomposed into the sum of externally applied fields and the internal self-consistent fields, produced by the particle motions. The internal fields are smoothed in space and time (variations that occur on very short time scales or over small distances are excluded). Equation (12) can be derived by considering the change of $f_\alpha$ if we follow the motion of some volume element in phase space. In this context the total time derivative of $f_\alpha$ is just an extension of the total time derivative of fluid mechanics. Equation (12) without the collision term is called the *Vlasov equation*. It is often not a bad approximation to set the collision term equal to zero, since a significant number of effects of the particle interactions is already included in the Lorentz force.

![Figure 3: The motion of some volume element in phase space, showing particles entering or leaving the volume element, as a result of collisions during time interval $dt$ (adapted after [6]).](Image)

The macroscopic transport equations for mass, momentum and energy can be obtained by multiplying equation (12) with the appropriate power of $v$ times $m_\alpha$ and integrating over velocity. This is called taking moments of the Boltzmann equation. In some cases the moments can be evaluated without the knowledge of $C_{\alpha\sigma}$. This is due to the fact, that the collision operator has to satisfy certain constraints, namely [5]:

- Conservation of particles $^6$: 
  $$\int C_{\alpha\sigma}(f_\alpha)\,d^3v = 0 \quad (14)$$

- Conservation of momentum: 
  $$\sum_{\alpha,\sigma} m_\alpha v C_{\alpha\sigma}(f_\alpha)\,d^3v = 0 \quad (15)$$

- Conservation of energy: 
  $$\sum_{\alpha,\sigma} m_\alpha v^2 C_{\alpha\sigma}(f_\alpha)\,d^3v = 0. \quad (16)$$

$^5$We only consider elastic (Coulomb) collisions which play a dominant role in fusion plasmas.

$^6$Here we assume that particles cannot be created or annihilated by collisions.
3.2 The equilibrium state

An equilibrium state is characterized by a time-independent solution of the Boltzmann equation or alternatively the maximum entropy of a closed system. The distribution function corresponding to a state of equilibrium should also be independent of \( \mathbf{r} \). The equilibrium distribution function in the absence of any external forces is the Maxwellian distribution \([6]\)

\[
f_{M\alpha}(\mathbf{v}) = n_{\alpha 0} \left( \frac{m_\alpha}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m_\alpha (\mathbf{v} - \mathbf{u}_\alpha)^2}{2k_B T} \right),
\]

where \( n_{\alpha 0}, m_\alpha, T \) and \( \mathbf{u}_\alpha \) are the particle density, mass, temperature and mean species velocity, respectively.

A state of local equilibrium \([7]\) is described by any function that is not necessarily independent of position and time, but for which the collision term in (12) vanishes. The distribution obeying such conditions is known as the local Maxwell distribution \([6]\)

\[
f_{M\alpha}(\mathbf{r}, \mathbf{v}, t) = n_\alpha(t) \left( \frac{m_\alpha}{2\pi k_B T(\mathbf{r}, t)} \right)^{3/2} \exp \left( -\frac{m_\alpha (\mathbf{v} - \mathbf{u}_\alpha(t))^2}{2k_B T(\mathbf{r}, t)} \right).
\]

Parameters \( n_\alpha, T \) and \( \mathbf{u}_\alpha \) now vary spatially and with time. The local Maxwell distribution is just an approximation, since it is in general not a solution of the Boltzmann equation \([7]\).

4 The two-fluid equations

As mentioned in the previous chapter, it is possible to obtain differential equations relating the macroscopic variables of each particle species \([8]\) by taking moments of the Boltzmann equation. For the "zeroth" moment we multiply equation (12) by \( m_\alpha \) and integrate over velocity \([9]\)

\[
\int m_\alpha \left[ \frac{\partial f_\alpha}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v} f_\alpha) + \frac{1}{m_\alpha} \frac{\partial}{\partial \mathbf{v}} \cdot \left( \mathbf{F} f_\alpha \right) \right] d^3 \mathbf{v} = \sum_\sigma \int m_\alpha C_{\alpha \sigma}(f_\alpha) d^3 \mathbf{v}.
\]

The condition (14) implies that the right-hand side of the above expression is zero. From the left-hand side we obtain the continuity equation

\[
\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha) = 0.
\]

\( \mathbf{u}_\alpha \) and \( \rho_\alpha \) are the mean velocity and mass density of species \( \alpha \). By multiplication with \( m_\alpha \mathbf{v} \) and integration, we obtain an equation of motion for each species \([5]\)

\[
n_\alpha m_\alpha \left( \frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha \right) = n_\alpha \epsilon_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) - \nabla \cdot \overleftarrow{P}_\alpha - \mathbf{R}_{\alpha \sigma}.
\]

\( \overleftarrow{P}_\alpha \) is the pressure tensor and is defined as \([5]\)

\[
\overleftarrow{P}_\alpha = m_\alpha \int \mathbf{v}' \otimes \mathbf{v}' f_\alpha \, d^3 \mathbf{v}',
\]

where \( \mathbf{v}' \) is the random part of a given velocity (that part of the velocity which differs from \( \mathbf{u}_\alpha \)). \( \mathbf{R}_{\alpha \sigma} \) accounts to the transfer of momentum between species \( \alpha \) and \( \sigma \) due to collisions. The collision terms have the form \([5]\)

\[
\mathbf{R}_{\alpha \epsilon} = \nu_{\alpha \epsilon} m_\epsilon n_\epsilon (\mathbf{u}_\epsilon - \mathbf{u}_\epsilon), \quad \mathbf{R}_{\epsilon \alpha} = \nu_{\epsilon \alpha} m_\epsilon n_\epsilon (\mathbf{u}_\epsilon - \mathbf{u}_\epsilon).
\]

\( \nu_{\alpha \epsilon} \) is the collision frequency of species \( \alpha \) with \( \epsilon \).
\( \nu_{\alpha\sigma} \) presents the rate at which the momentum of species \( \alpha \) is destroyed due to collisions with species \( \sigma \).

The macroscopic equations presented above are much easier to solve than the Boltzmann equation and the moments represent more useful information, since they describe measurable physical quantities. However, there is a certain price for such greater simplicity. It turns out that this moment-taking procedure never leads to a closed set of equations and one always ends up with more unknowns than equations. Therefore it is necessary to introduce some simplifying assumptions concerning the highest moment of the distribution function in our set of equations. For example, if we work with the equation of motion and equation of continuity, we have to make an assumption about the form of the pressure tensor \( \mathbf{P} \) and relate it’s components with other macroscopic variables. A typical closure of this type would involve invoking adiabatic or isothermal assumptions.

5 Magnetohydrodynamics

Magnetohydrodynamics is an alternate description of the plasma where instead of using \( u_e \) and \( u_i \) to describe mean motion, two new velocity variables that are a linear combination of \( u_e \) and \( u_i \) are used. The new velocity-like variables are the electric current density

\[
j = \sum_{\alpha} n_{e\alpha} e_{\alpha} u_{\alpha}, \tag{24}\]

which is essentially the relative velocity between ions and electrons, and the center of mass velocity

\[
u = \sum_{\alpha} \frac{1}{\rho} m_{\alpha} n_{\alpha} u_{\alpha}, \tag{25}\]

where

\[
\rho = \sum_{\alpha} m_{\alpha} n_{\alpha} \tag{26}\]

is the total mass density. Since ions are much heavier than electrons, the center of mass velocity is usually close to the ion velocity \( \nu \approx u_i \). In the above representation, the plasma is considered as a single, conducting fluid. Magnetohydrodynamics is primarily concerned with low-frequency, long-wavelength, magnetic behavior of plasma.

5.1 Equation of motion

The MHD equation of motion can be derived by taking the first moment of the Vlasov equation and summing over all particle species, so that the result can be expressed in terms of the quantities \( \nu \) and \( j \). The result is [5]

\[
\rho \frac{D\nu}{Dt} = j \times B - \nabla \cdot \mathbf{\hat{P}}, \tag{27}\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + (\nu \cdot \nabla) \]

is the (hydrodynamic) convective derivative. \( \mathbf{\hat{P}} \) is the MHD pressure tensor, defined in terms of the random velocities relative to \( \nu \) and is given by [5]

\[
\mathbf{\hat{P}} = \sum_{\alpha} m_{\alpha} \int \nu' \otimes \nu' f_{\alpha} d^3 \nu. \tag{28}\]

The only assumption of (27) is, that the characteristic length of phenomena under consideration is much larger than the Debye length \( \lambda_D \) and characteristic frequency much smaller than the plasma frequency \( \omega_p \). As a consequence, the plasma may be considered charge neutral

\[
\sum_{\alpha} n_{\alpha} e_{\alpha} \approx 0, \tag{29}\]

so that the electric field \( E \) cannot exert any force on the fluid. Condition (29) requires that the electron and ion number densities are approximately the same \( n_i \approx n_e = n \).
5.2 Electrodynamic equations

The electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are as always determined by the Maxwell equations. A very common step at this point, found in many textbooks on plasma physics, is to neglect the displacement current. This assumption is well justified, because we are interested only in the low-frequency, long-wavelength phenomena. Formally, the neglect of displacement current requires that the electromagnetic waves of interest have phase velocities much slower than the speed of light $\omega/k \ll c$ and that the thermal velocities of ions and electrons be nonrelativistic $v_T \ll c$ [8]. Making use of the above assumptions, the Maxwell equations may be written as

$$\nabla \cdot \mathbf{E} = \sum \frac{n_e \mathbf{v}_e}{\epsilon_0}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{j} + \mathbf{j}_{\text{ext}}) + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}.$$  \hfill (30)

(31)

$j_{\text{ext}}$ is the current density produced by currents external to the plasma (for example, it may represent the electric currents flowing in the magnetic coils of a fusion device).

5.3 The generalized Ohm’s law

The MHD equation of motion provides a relation between $\mathbf{j}$ and $\mathbf{u}$. It is possible to obtain a second relation from the 2-fluid equation for electrons (21). Because $m_e \ll m_i$, the electron response time is essentially instantaneous. This fact suggests, that we may drop the electron electron inertia term $m_e \frac{\partial \mathbf{u}_e}{\partial t}$ on the time scale of interest. In order to do this, the characteristic length $L$ and frequency $\omega$ have to satisfy the following conditions [9]:

- $\omega \ll \omega_p, \lambda_D \ll L$ (no plasma oscillations $\parallel$ to $\mathbf{B}$),
- $\omega \ll \Omega_c = \frac{eB}{m_e}, v_{Ti}/\Omega_c = r_L \ll L$ (no electron cyclotron oscillations).

$\Omega_c$ is the electron cyclotron frequency with $r_L$ as the radius of gyration. $v_{Ti}$ is the electron thermal velocity. The approximation with the acceleration term equal to zero is the most severe one made so far. It is often not very well justified, since the velocity component parallel to $\mathbf{B}$ is treated badly in this approximation. However, in most MHD problems the parallel motion doesn’t play an important role, so that the theory is still able to deliver accurate results. Because $m_e \ll m_i$ we have $\mathbf{u}_e \approx \mathbf{u} - \mathbf{j}/n_e$, so that equation (21) reduces to the generalized Ohm’s law:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{n_e} \mathbf{j} \times \mathbf{B} + \frac{1}{n_e} \nabla \cdot \mathbf{P_e} = \eta \mathbf{j},$$ \hfill (32)

where $\eta$ is the specific resistivity

$$\eta = \frac{m_e \nu_{ei}}{n_e}.$$ \hfill (33)

The term $\frac{1}{n_e} \mathbf{j} \times \mathbf{B}$ is called the Hall term. It may be dropped if the pressure is small or if the electron-ion collision frequency is large compared to the electron cyclotron frequency.

5.4 Isothermal limit

The most simplistic assumption leading to a closed set of equations is the isothermal limit, where we consider a plasma in thermal equilibrium with constant temperature $T$. Since the plasma is in equilibrium, the velocity distribution is the Maxwellian, so that the pressure is fully isotropic and the equation of state is given by $p = k_B n T$. The $\nabla p$ term in the equation of motion can therefore be written as [10]

$$\nabla p = k_B T \nabla n.$$ \hfill (34)

10This is due to the fact that parallel velocities do not provide a magnetic force.
5.5 Adiabatic limit

In the adiabatic limit, we set the plasma heat conductivity to zero, so that there can be no heat exchange across the system. This is a reasonable assumption, if for example, the electron mean free path is much smaller than the characteristic length of phenomena under observation. The adiabatic regime is described by the relation \[ \frac{p}{\rho \gamma} = \text{const.}, \]
where \( \gamma \) is the ratio of specific heats at constant pressure and at constant volume. If \( N \) is the number of degrees of freedom, \( \gamma \) is given by
\[
\gamma = \frac{2 + N}{N}. \tag{36}
\]
For the case of a monatomic gas with no internal degrees of freedom, we have \( N = 3 \) and therefore \( \gamma = \frac{5}{3} \).

The \( \nabla p \) term is given by [6]
\[
\nabla p = \frac{\gamma p}{\rho} \nabla \rho. \tag{37}
\]

5.6 Ideal MHD

Ideal MHD is a limiting regime, obtained by neglecting the viscosity of the fluid, thermal conductivity, specific resistivity and assuming that the pressure is isotropic. Since ideal MHD equations are easier to solve, they can be used to investigate realistic geometries. The theory has thereby provided a useful and surprisingly accurate description of the macroscopic behavior of fusion plasmas, showing why certain field configurations are more favorable to confinement than others.

The basic requirement for the validity of ideal MHD is, that both the electrons and ions be collision dominated [8]. This makes the pressure isotropic. In this limit, the distribution function for both electrons and ions is nearly locally Maxwellian. The condition that each species be collision dominated is given by [8]

- Ions: \( v_{T_i} \tau_{ii} / L \ll 1 \),
- Electrons: \( v_{T_e} \tau_{ee} / L \ll 1 \),

where \( v_{T_i} \) and \( v_{T_e} \) are the ion and electron thermal velocities \( (v_{T_{\alpha}} = (2k_B T/m_{\alpha})^{1/2}) \) and \( \tau_{ii}, \tau_{ee} \) the ion-ion and electron-electron collision times. For fusion plasmas, we have \( L \sim 1 \text{ m} \), \( \tau_{ee} \approx \tau_{ii} \left( \frac{m_e}{2m_i} \right)^{1/2} \sim 10 \text{ µs} \), \( v_{T_e} \sim 0.09 \text{ c}, v_{T_i} \sim 0.002 \text{ c} \) and \( v_{T_e} \tau_{ee} / L \approx v_{T_i} \tau_{ii} / L \sim 270 \text{ m} \). This clearly doesn’t fulfill the above condition, although large empirical evidence shows that MHD gives a very accurate description of a fusion plasma equilibrium. Some of the reasons for this apparent contradiction are mentioned at the end of this section.

<table>
<thead>
<tr>
<th>Plasma physics time scales</th>
<th>Formulas</th>
<th>Numerical values (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron gyro period</td>
<td>( \tau_{\text{gyro}} = 2\pi/\omega_{\text{gyro}} = 2\pi m_e/eB_0 )</td>
<td>( 7.1 \times 10^{-12} )</td>
</tr>
<tr>
<td>Electron plasma period</td>
<td>( \tau_{\text{pe}} = 2\pi/\omega_{\text{pe}} = 2\pi (m_e e^2 / ne^2)^{1/2} )</td>
<td>( 7.9 \times 10^{-12} )</td>
</tr>
<tr>
<td>Ion plasma period</td>
<td>( \tau_{m} = (m_i/m_e)^{1/2} \tau_{\text{pe}} )</td>
<td>( 4.8 \times 10^{-10} )</td>
</tr>
<tr>
<td>Ion gyro period</td>
<td>( \tau_{ii} = (m_i/m_e) \tau_{\text{gyro}} )</td>
<td>( 2.6 \times 10^{-8} )</td>
</tr>
<tr>
<td>MHD time</td>
<td>( \tau_{m} = \frac{a}{V} )</td>
<td>( 2.3 \times 10^{-8} )</td>
</tr>
<tr>
<td>Electron–electron collision time</td>
<td>( \tau_{ee} = 7.4 \times 10^{-6} \tau_{e}^2 / n )</td>
<td>( 1.0 \times 10^{-5} )</td>
</tr>
<tr>
<td>Ion–ion collision time</td>
<td>( \tau_{\text{eq}} = (2m_i/m_e)^{1/2} \tau_{\text{ee}} )</td>
<td>( 8.9 \times 10^{-4} )</td>
</tr>
<tr>
<td>Energy equilibration time</td>
<td>( \tau_{eq} = (m_i/2m_e) \tau_{\text{ee}} )</td>
<td>( 1.9 \times 10^{-5} )</td>
</tr>
<tr>
<td>Ignition time</td>
<td>( \tau_{\text{ign}} = 2.0 / n )</td>
<td>1.0</td>
</tr>
<tr>
<td>Resistive diffusion time</td>
<td>( \tau_{\text{D}} = \mu a^2 / \eta = 51a^2 \tau_{e}^2 )</td>
<td>( 1.4 \times 10^2 )</td>
</tr>
</tbody>
</table>

Table 1: Comparison of the characteristic MHD time with that of other plasma physics phenomena. Ideal MHD lies midway between a variety of high-frequency microscopic phenomena and the very low-frequency transport phenomena. This is the regime of macroscopic equilibrium and stability [8].

There are also other conditions for the validity of ideal MHD. Here, we only give a list of requirements, without the explicit form of expressions that represent these conditions:
• The ion gyro radius is very small compared to the characteristic length (the terms \( \frac{1}{\omega_e} j \times B \) and \( \nabla \cdot \vec{P}_e \) in equation (32) may be dropped);

• The size of the plasma is large, so that resistive diffusion doesn’t play an important role (\( \eta j \approx 0 \));

• Short energy equilibration time (\( T_i \approx T_e \));

A complete analysis of the criteria listed above can be found in ([8], Chapter 2). The full set of ideal MHD equations can now be written as

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\n\partial \rho = \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\mathbf{E} + \mathbf{u} \times \mathbf{B} &= 0 \\
\rho \frac{D \mathbf{u}}{Dt} &= \mathbf{j} \times \mathbf{B} - \nabla p \\
\frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) &= 0.
\end{align*}
\]

As already mentioned, it turns out that typical fusion plasma parameters are outside the region of validity for MHD, but large empirical evidence shows that the theory is very accurate in describing the macroscopic equilibrium and stability of fusion plasmas. The reason for this is the fact, that most errors appear in the motions parallel to \( \mathbf{B} \). As already mentioned, these motions don’t have a strong influence on the macroscopic equilibrium of fusion plasmas.

6 Tokamaks

Tokamaks are nowadays the most promising devices for the development of nuclear fusion power plants. These devices could be one of the best approaches to energy generation in the long term and therefore are the subject of intensive international programmes of research, one of the most important being the ITER project. In a tokamak, the plasma is confined inside a doughnut-shaped chamber with the use of strong magnetic fields.

Some of the main components of the machine are [11]:

• the vacuum vessel, the chamber where the plasma is contained;

• the blanket, which has the main roles of absorbing the 14 MeV neutrons and of breeding the plasma with tritium, which is needed for the reaction. For this reason, the blanket is usually composed of lithium;

• the toroidal coils. The currents flowing in these coils generate a poloidal field (poloidal field coils, or PF coils). The coils in the center of the machine are called central solenoids (CS); the plasma current is built up by transformer action, where the central solenoids are the transformer primary and the plasma itself acts as the transformer secondary;

• the poloidal coils. The currents flowing in these coils generate a toroidal field (toroidal field coils, or TF coils).

---

11 ITER stands for International Thermonuclear Experimental Reactor. The experimental site in Cadarache, France is currently under construction.

12 The plane, characterized by a constant toroidal angle \( \phi \) in cylindrical coordinates is called the poloidal plane and the projection of \( \mathbf{B} \) onto this plane is the poloidal field.
6.1 Principles of magnetic confinement in tokamaks

Magnetic confinement is in general associated with a state of the macroscopic MHD fluid in which all identified forces are exactly in balance providing thereby a time-independent solution. The temperatures needed for fusion ignition are of the order of a 100 million degrees, meaning that if the plasma was to contact a material surface, it would immediately cool down and damage the surface. Therefore, there is no solid container that can work. A general property of any perfectly conducting fluid is the conservation of magnetic flux in the frame moving with the fluid velocity $u$. The fact that the field lines are "frozen" into the plasma leads naturally to the notion that by controlling the magnetic field configuration one might be able to contain the fluid. The geometry of the magnetic field lines, needed for efficient confinement of tokamak plasmas, is usually investigated for the case of plasmas with a near-Maxwellian distribution in the ideal MHD limit. Issues requiring a 2-fluid or a kinetic description can exist, but these questions can be addressed after an approximate understanding has first been achieved using MHD.

Let us now investigate the magnetic configuration in static equilibrium with the fluid velocity $u$ equal to zero, since this is the case of real tokamak devices. From the ideal MHD equation of motion we obtain the relation

$$\nabla p = j \times B,$$

which is equivalent to the statement $\nabla p \cdot B = 0$. From this it follows, that the magnetic field lines have to lie in the surfaces of constant pressure. These surfaces can be spatially bounded only if they have the topological form of a torus. The isobaric surfaces in a tokamak equilibrium are therefore toroidally nested. As a consequence of the fact that the magnetic field lines lie on the isobaric surfaces, these surfaces are also called magnetic surfaces. The limiting magnetic surface, which approaches a single magnetic line, where the pressure is maximum, is called the magnetic axis.

---

13The surface is also damaged by the neutron flux produced in fusion reactions and by the electromagnetic radiation. This kind of damage to the material is not so severe as the one caused by a direct contact with the plasma, although it is true, that during a long-term operation of a tokamak, the blanket would have to be eventually replaced, as a result of material damage, caused by neutrons.

14This is a consequence of a topological theorem, which states that a non-singular vector field $B$ can be everywhere tangent to a spatially bounded function $p$ only in the shape of a torus [3].

---
There is another possible interpretation of the equation (38), which can be alternatively written as

$$\nabla p = \frac{1}{\mu_0} (\nabla \times B) \times B. \quad (39)$$

The term $\frac{1}{\mu_0} (\nabla \times B) \times B$ may be presented as a divergence of the magnetic stress tensor $T_{ij}^{(m)}$. The magnetic stress tensor is given by [6]

$$T_{ij}^{(m)} = \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right). \quad (40)$$

By considering the relation introduced above, the condition (39) may also be given as

$$\nabla \cdot \left( pI - \frac{T^{(m)}}{\mu_0} \right) = 0. \quad (41)$$

In a local coordinate system, with the z-axis aligned with B we have

$$\nabla \cdot \begin{pmatrix} (p + B^2/2\mu_0) & 0 & 0 \\ 0 & (p + B^2/2\mu_0) & 0 \\ 0 & 0 & (p - B^2/2\mu_0) \end{pmatrix} = 0. \quad (42)$$

The above equation has a very illustrative meaning. First, we introduce the so called magnetic pressure $p_M = B^2/2\mu_0$. When the curvature of B and it’s variation along the field lines are small [4], the sum of kinetic and magnetic pressure has to remain approximately constant throughout the plasma. The kinetic pressure decreases from the magnetic axis radially outwards, whereas the magnetic field increases in the same direction in such manner that their sum remains constant. At the plasma boundary we should have $p = 0$ and therefore [4]

$$p + B^2/2\mu_0 \sim B_0^2/2\mu_0, \quad (43)$$

where $B_0$ is the magnitude of the field at the plasma boundary. If we want the plasma to be confined by the magnetic field, at least the relation

$$\beta = \frac{p}{B_0^2/2\mu_0} < 1 \quad (44)$$

should hold. The quantity $\beta$ is known as the beta ratio and is a very important characteristic of a tokamak device. Fusion power density varies as $\beta^2$ so that high $\beta$ values are desirable recognizing, however a limitation due to plasma instabilities.

### 6.1.1 The Grad-Shafranov equation

For a plasma with an axisymmetric shape, the equilibrium profiles of the isobaric and magnetic surfaces are determined by the Grad-Shafranov equation.
The first step towards the Grad-Shafranov equation is to write the MHD equations in cylindrical coordinates \((r, \phi, z)\) and assume that any quantity may only depend on \(r\) and \(z\). The next step is to introduce the so called *poloidal flux function*

\[
\psi(r) = \int_0^r r' B_z(r', z) \, dr',
\]  
(45)

which represents the magnetic flux through a circular surface, perpendicular to the \(z\) axis, with it’s center at \(r = 0\), as shown in figure 7.

![Figure 7](Image)

**Figure 7**: The cylindrical coordinate system \((r, \phi, z)\) with the plasma poloidal cross-section (gray area) and the magnetic flux, denoted by the function \(\psi\).

The poloidal magnetic field \(B_p\) is linked with the flux function by the relation [11]

\[
B_p = \nabla \psi \times \nabla \phi.
\]  
(46)

The fact, that we are able to determine \(B_p\) by it’s own integral over some surface is made possible by the additional condition \(\nabla \cdot B = 0\). Since we also have \(\nabla \cdot j = 0\), we may present the poloidal current density \(j_p\) in the same way as \(B_p\). \(j_p\) is therefore given by

\[
j_p = \nabla f \times \nabla \phi, \tag{47}
\]

where the definition of function \(f(r)\) is analogous to that of \(\psi\). Finally, combining equations (46) and (47) with \(\nabla \times B = \mu_0 j\) we get [11]

\[
B = \nabla \psi \times \nabla \phi + \mu_0 f \nabla \phi \tag{48}
\]

\[
j = \nabla f \times \nabla \phi + \mu_0^{-1} \Delta^* \psi \nabla \phi, \tag{49}
\]

where \(\Delta^*\) is the differential elliptic operator defined as

\[
\Delta^* = r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2}.
\]

The poloidal flux function has another important property, namely

\[
B \cdot \nabla \psi = 0 \tag{50}
\]

The magnetic or isobaric surfaces are also surfaces of constant \(\psi\). We can therefore consider \(p\), \(B\) and \(j\) as functions of \(\psi\) only. With this assumption the force equilibrium equation (38) can be rewritten in the form [11]

\[
\Delta^* \psi = -\mu_0 r^2 \frac{d}{d\psi} p - \mu_0^2 f \frac{d}{d\psi} f. \tag{51}
\]

This is the famous Grad-Shafranov equation. It can be solved once the current density in the magnetic coils and passive structures of the machine, as well as the functions \(p(\psi)\) and \(f(\psi)\), have been assigned.
Figure 8: Constant level curves of the poloidal flux function for a plasma equilibrium in a tokamak device. The thicker line corresponds to the value of the poloidal flux at the plasma boundary. The values of $\psi(r)$ were obtained by numerical solution of the Grad-Shafranov equation [11].

7 Conclusions

The magnetohydrodynamic theory constitutes a very important part for the understanding of basic fusion plasma equilibrium. Without the relatively simple form of MHD equations, it would be almost impossible to investigate plasma confinement in realistic geometries. Because instabilities in fusion plasmas can never be completely avoided, the plasma has to be carefully controlled by means of feedback stabilization. Plasma magnetic control is actually one of the main challenges for a successful operation of tokamaks in the future. So far, the implemented controller schemes have relied heavily on the MHD equations, especially the Grad-Shafranov equation. We therefore suspect, that MHD theory will keep it’s important role in the field of fusion power research and will provide an even better understanding of plasma magnetic confinement.

References


