Abstract

In this paper I will present various types of single particle motion in externally applied electro-magnetic fields, where they arise in nature and engineering, and discuss the applicability of such a model for the representation of plasmas.
# Contents

1 Introduction 3  

2 Uniform B field 3  
  2.1 $E = 0$ ......................................................... 3  
  2.2 Constant $E$ .................................................. 4  
  2.3 Additional force $F$ ......................................... 6  

3 Non-uniform B field 6  
  3.1 $\nabla B$ drift ................................................. 6  
  3.2 Curvature drift ............................................. 8  
  3.3 Magnetic mirror ............................................ 11  

4 Time-varying E field 12  
  4.1 Polarization drift .......................................... 12  
  4.2 Cyclotron resonance ...................................... 13  

5 Conclusion 14  

References 15
1 Introduction

Roughly 99% of matter in the universe is in the plasma state - a gas composed of neutral and electrically charged particles. Stars, nebulae, and most of the interstellar hydrogen are all plasma. And yet we rarely encounter naturally-occurring plasma in our everyday lives - almost exclusively in the form of lightning bolts or the Aurora Borealis. This is explained by the Saha equation, which relates the amount of ionization in a gas at thermal equilibrium to its temperature

$$\frac{n_i}{n_n} \approx 2.4 \times 10^{21} \frac{T^{3/2}}{n_i} e^{-U_i/KT}$$

where $n_i$ and $n_n$ are the number densities of ionized and neutral atoms, $U_i$ is the ionization energy of the gas, $T$ is the temperature and $K$ is the Boltzmann constant. For air at room temperature, this gives an $n_i/n_n$ ratio of $10^{-122}$, explaining why we so seldom encounter plasma [2].

To understand the behavior of a plasma it is necessary to first understand the motion of a single charged particle in an externally imposed electro-magnetic (EM) field. While such an approach does not explicitly touch upon the collective behavior exhibited by plasmas, it is a useful first approximation. It can reveal many faults in magnetic confinement devices, such as those used for nuclear fusion, and tells us about the behavior of space plasmas, such as the Van-Allen radiation belts in the Earth’s magnetosphere, which is relevant to artificial satellites, since the highly energetic plasma trapped in the belts can cause damage to both electronics and organic life.

Throughout this paper I will take $\mathbf{E}$ to be the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{v}$ the velocity, $m$ will be mass, and $q$ the electric charge. Vectors are written as boldface, such as $\mathbf{B}$, and we take the same non-boldface variable to mean the absolute value, so that $\mathbf{B} = |\mathbf{B}|$. We will also adopt the convention that $\mathbf{A} \cdot \mathbf{B} = AB$.

2 Uniform B field

2.1 $\mathbf{E} = 0$

In this case a particle has a simple cyclotron gyration. The equation of motion is

$$\dot{\mathbf{v}} = \frac{q}{m} \mathbf{v} \times \mathbf{B}$$

We can assume $\mathbf{B} = (0, 0, B)$, which yields

$$\begin{align*}
\dot{v}_x &= \frac{qB}{m} v_y \\
\dot{v}_y &= -\frac{qB}{m} v_x \\
\dot{v}_z &= 0 \\
\dot{v}_x &= \frac{qB}{m} \dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x \\
\dot{v}_y &= -\frac{qB}{m} \dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y
\end{align*}$$

With the exception of describing velocity, not position, these equations are otherwise identical to those of a harmonic oscillator, with a frequency of

$$\omega_c = \frac{qB}{m}$$
also known as the angular cyclotron frequency [3]. Typical cyclotron frequencies inside a tokamak are a few tens of GHz for electrons and a few tens of MHz for ions. The solution will be of the form

\[
\begin{align*}
    v_x &= a \cos(\omega_c t + \delta_x) \\
    v_y &= b \cos(\omega_c t + \delta_y)
\end{align*}
\]

Inserting eq. 2.3 into eq. 2.1, we find that \(a = b = v_\perp\) and \(\delta_x = \delta_y + \frac{\pi}{2}\). Setting \(t\) to 0, we divide eq. 2.4 by eq. 2.3, which yields \(v_y(0)/v_x(0) = -\tan \delta\), where \(v_x(0) = v_x(t = 0)\) and \(v_y(0) = v_y(t = 0)\). This gives us

\[
\begin{align*}
    \delta &= -\arctan \frac{v_y(0)}{v_x(0)} \\
    v_x &= v_\perp \cos(\omega_c t + \delta) \\
    v_y &= -v_\perp \sin(\omega_c t + \delta) \\
    v_z &= v_z(0)
\end{align*}
\]

Since \(v_x^2 + v_y^2 = v_\perp^2\), it follows that \(v_\perp\) is the velocity in the plane perpendicular to \(B\). To find the position, we simply integrate the above equations

\[
\begin{align*}
    x &= r_L \sin(\omega_c t + \delta) + x_c \\
    y &= r_L \cos(\omega_c t + \delta) + y_c
\end{align*}
\]

Where \(x_c\) and \(y_c\) are integrating constants derived from \(\delta\) and position at \(t = 0\), and

\[r_L = \frac{v_\perp}{\omega_c}\]

is the Larmor radius. These equations produce a guiding center motion around the point \((x_c, y_c, z)\), and a constant velocity parallel to \(B\). Particles thus follow a helical trajectory. The direction of gyration is always such that the magnetic field produced by the particle is opposite the external field. Plasmas are therefore diamagnetic [2].

Conversely, we can solve the above equations using vector analysis. We can write \(\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel\), where \(\mathbf{v}_\perp\) is perpendicular, and \(\mathbf{v}_\parallel\) is parallel, to \(B\). The equation of motion is thus split in two:

\[
\begin{align*}
    \dot{\mathbf{v}}_\parallel &= \frac{q}{m}(\mathbf{v}_\parallel \times \mathbf{B}) = 0 \\
    \dot{\mathbf{v}}_\perp &= \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B})
\end{align*}
\]

The first equation tells us that \(\mathbf{v}_\parallel\) is constant. The second equation tells us that \(\dot{\mathbf{v}}_\perp\) is always perpendicular to \(\mathbf{v}_\perp\). Thus \(\mathbf{v}_\perp\), and consequently \(|\mathbf{v}_\perp \times \mathbf{B}|\), are constant. This means that acceleration is constant in magnitude and always perpendicular to \(\mathbf{B}\) and \(\mathbf{v}_\perp\), and therefore describes circular motion. [4]

### 2.2 Constant E

We introduce a constant external field \(\mathbf{E} = (E_x, 0, E_z)\). The equation of motion is now

\[m\ddot{\mathbf{v}} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E})\]

Or separately for each component

\[
\begin{align*}
    \dot{v}_x &= \frac{q}{m}E_x + \omega_c v_y \\
    \dot{v}_y &= -\omega_c v_x \\
    \dot{v}_z &= \frac{q}{m}E_z
\end{align*}
\]
The solution for the z-component is simple acceleration

\[ v_z = \frac{qE_z}{m} t + v_{z0} \]

Taking the time derivative of 2.5 and 2.6 we get

\[
\begin{align*}
\ddot{v}_x &= \omega_c \dot{v}_y = -\omega_c^2 v_x \\
\ddot{v}_y &= -\omega_c \dot{v}_x = -\omega_c \left( \frac{q}{m} E_x + \omega_c v_y \right) \\
&= -\omega_c^2 \left( \frac{E_x}{B} + v_y \right)
\end{align*}
\]

The second equation is equivalent to

\[
\frac{d^2}{dt^2} \left( v_y + \frac{E_x}{B} \right) = -\omega_c^2 \left( v_y + \frac{E_x}{B} \right)
\]

If we replace \( v_y + E_x/B \) with \( v_y \), the above expression becomes identical to eq. 2.2. Thus the solution is

\[
\begin{align*}
v_x &= v_\perp \cos(\omega_c t + \delta) \\
v_y &= -v_\perp \sin(\omega_c t + \delta) - \frac{E_x}{B}
\end{align*}
\]

In addition to the previous motion, we now have an additional drift parallel to \( \hat{y} \). To find the direction of this drift in general, let us recall that \( \mathbf{B} = (0, 0, B) \) and \( \mathbf{E} = (E_x, 0, E_z) \), meaning that this drift is perpendicular to both \( \mathbf{B} \) and \( \mathbf{E} \), meaning it is parallel to \( \mathbf{E} \times \mathbf{B} = (0, -E_x B, 0) \). If we merely divide this by \( B^2 \), we obtain

\[
\mathbf{v}_{E\times B} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = (0, -\frac{E_x}{B}, 0)
\]

which we shall call the \( \mathbf{E} \times \mathbf{B} \) drift [2], and is exactly the additional velocity introduced by \( \mathbf{E} \) in the direction perpendicular to \( \mathbf{B} \). Since we can transform any reference frame into one where
\( \mathbf{B} \parallel \mathbf{\hat{z}} \) and \( \mathbf{E} \perp \mathbf{\hat{y}} \), it holds that the vector form of the above equation is always applicable (for uniform and constant \( \mathbf{B} \) and \( \mathbf{E} \)).

Thus the motion of a charged particle in this case is that of a slanted spiral - the particle circles a guiding center, which is accelerated in the \( \mathbf{\hat{z}} \) direction by \( \mathbf{E} \), and moves at a velocity of \( \mathbf{v}_E \) in the \( \mathbf{\hat{y}} \) direction. Note that \( \mathbf{v}_E \) does not depend on particle mass, or even charge, meaning both positively and negatively charged particles will drift in the same direction.

### 2.3 Additional force \( \mathbf{F} \)

If we introduce an additional force \( \mathbf{F} \) of arbitrary origin, the equation of motion now reads

\[
\mathbf{m} \ddot{\mathbf{v}} = q(\mathbf{v} \times \mathbf{B}) + \mathbf{F} = q(\mathbf{v} \times \mathbf{B} + \frac{\mathbf{F}}{q})
\]

Thus the additional drift caused by \( \mathbf{F} \) can be obtained by replacing \( \mathbf{E} \) with \( \frac{\mathbf{F}}{q} \) in eq. 2.7, which yields

\[
\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{q \mathbf{B}^2}
\]

(2.8)

Note that unlike \( \mathbf{v}_E \), \( \mathbf{v}_F \) depends upon the charge \( q \), and thus points in opposite directions for positively and negatively charged particles, which results in a net current and subsequent charge separation. \( \mathbf{F} \) can, among other things, represent a systemic force in a non-inertial reference frame.

If it represents gravity it is called the *gravitational drift* [1], in which case its magnitude is on the order of \( 10^{-11} \) m/s in typical fusion setups, and \( 10^{-5} \) m/s in the Earth’s Van-Allen radiation belts, and is in both cases negligible compared to the curvature and gradient drifts.

### 3 Non-uniform \( \mathbf{B} \) field

#### 3.1 \( \nabla \mathbf{B} \) drift

To calculate the motion in a non-uniform magnetic field, it is necessary to make some approximations. We shall assume that the inhomogeneity of our field is small on scales of \( r_L \), so that
\[ \| \mathbf{B}(\mathbf{r}_0 + \mathbf{r}_1) - \mathbf{B}(\mathbf{r}_0) \| \ll B(\mathbf{r}_0) \] for any \( r_1 \leq r_L \). We write
\[
\begin{align*}
\mathbf{r} &= \mathbf{r}_0 + \mathbf{r}_1 \\
\mathbf{v} &= \mathbf{v}_0 + \mathbf{v}_1 \\
\mathbf{v}_0 &= \dot{\mathbf{r}}_0, \mathbf{v}_1 &= \dot{\mathbf{r}}_1
\end{align*}
\]
where \( \mathbf{r}_0 \) and \( \mathbf{v}_0 \) describe simple guiding center motion in a homogenous magnetic field, and \( \mathbf{r}_1 \) and \( \mathbf{v}_1 \) are small perturbations due to inhomogeneity. Next we split \( \mathbf{B} \) into two parts
\[
\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1
\]
Where \( \mathbf{B}_0 = (0, 0, B_{0z}) \) is the main part, and \( \mathbf{B}_1 \) is a small disturbance - the source of our inhomogeneity. The equation of motion becomes
\[
m\dot{\mathbf{v}} = q(\mathbf{v} \times \mathbf{B})
\]
\[
= q((\mathbf{v}_0 + \mathbf{v}_1) \times (\mathbf{B}_0 + \mathbf{B}_1))
\]
\[
= q(\mathbf{v}_0 \times \mathbf{B}_0) + q(\mathbf{v}_0 \times \mathbf{B}_1) + q(\mathbf{v}_1 \times \mathbf{B}_0) + q(\mathbf{v}_1 \times \mathbf{B}_1)
\]
Since \( r_1 \ll r_0 \) and \( v_1 \ll v_0 \), the right-most term in the above equation is a second-order correction and can be discarded. Joining the 1\textsuperscript{st} and 3\textsuperscript{rd} term then yields
\[
m\dot{\mathbf{v}} \approx q(\mathbf{v}_0 \times \mathbf{B}_0) + q(\mathbf{v}_0 \times \mathbf{B}_1) + q(\mathbf{v}_1 \times \mathbf{B}_0)
\]
\[
\approx q(\mathbf{v} \times \mathbf{B}_0) + q(\mathbf{v}_0 \times \mathbf{B}_1)
\]
Moving into a stationary reference frame where the guiding center is at \( \mathbf{0} \) at time 0, and setting \( \mathbf{B}_0 \) to be the total field at \( \mathbf{0} \) means \( \mathbf{B}(\mathbf{0}) = \mathbf{B}_0 \) and \( \mathbf{B}_1(\mathbf{0}) = \mathbf{0} \), thus we may write
\[
\mathbf{B}_1(\mathbf{r}) \approx (\mathbf{r} \nabla) \mathbf{B}
\]
We introduced the notation \( \mathbf{B}(\mathbf{0}) = \mathbf{B} \). This gives the equation
\[
m\dot{\mathbf{v}} \approx q(\mathbf{v} \times \mathbf{B}_0) + q(\mathbf{v}_0 \times (\mathbf{r} \nabla) \mathbf{B})
\]
\[
\approx q(\mathbf{v} \times \mathbf{B}_0) + \mathbf{F}(\mathbf{r}, \mathbf{v}_0)
\]
Which is very similar to 2.8, except that now \( \mathbf{F} \) is a function of position and velocity. It is possible to further simplify the expression for \( \mathbf{F} \):
\[
\mathbf{F} = q(\mathbf{v}_0 \times (\mathbf{r} \nabla) \mathbf{B})
\]
\[
= q(\mathbf{v}_0 \times ((\mathbf{r}_0 + \mathbf{r}_1) \nabla) \mathbf{B})
\]
\[
= q(\mathbf{v}_0 \times ((\mathbf{r}_0 \nabla) \mathbf{B}) + q(\mathbf{v}_0 \times ((\mathbf{r}_1 \nabla) \mathbf{B})
\]
\[
\approx q(\mathbf{v}_0 \times ((\mathbf{r}_0 \nabla) \mathbf{B})
\]
In the last step we dropped the final term, since it is of second-order. Thus \( \mathbf{F} \) becomes a function of only \( \mathbf{r}_0 \) and \( \mathbf{v}_0 \). To use equation 2.8, we will average \( \mathbf{F} \) over a cyclotron gyration. First we define \( \mathbf{r}_0 \) and \( \mathbf{v}_0 \):
\[
\begin{align*}
\mathbf{r}_0 &= (x_0, y_0, z_0) & \mathbf{v}_0 &= \dot{\mathbf{r}}_0 \\
x_0 &= r_L \sin (\omega_c t + \varphi) \\
y_0 &= r_L \cos (\omega_c t + \varphi) \\
z_0 &= v_{z0} t
\end{align*}
\]
Figure 3: $\nabla B$-drift of a positive and negative particle [2].

We used the total field at $\vec{0}$, $B$, to calculate $\omega_c$. Next we average over a cyclotron gyration

$$F_{avg} = \frac{1}{t_0} \int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} F dt = \frac{mv_\perp}{2B} \left( \begin{array}{c} -\partial_x B_z v_\perp - 2\partial_z B_z v_\perp \cos \varphi \\ \partial_y B_z v_\perp - 2\partial_z B_z v_\perp \sin \varphi \\ -\partial_z B_z v_\perp + 2\partial_z B_z v_\perp \cos \varphi - 2\partial_z B_y v_\perp \sin \varphi \end{array} \right)$$

Where $t_0 = 2\pi/\omega_c$, and we used $\nabla B = 0$ to replace $\partial_x B_x + \partial_y B_y$ with $-\partial_z B_z$. This has given us a force that is dependent upon the arbitrary angle $\varphi$, which is unexpected. Since a particle’s initial angle $\varphi$ is equally likely to have any value between 0 and $2\pi$ we again average over $\varphi$:

$$F_{avg} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{avg1} d\varphi = -\frac{W_\perp}{B} \frac{1}{B} \left( \begin{array}{c} \partial_x B_z \\ \partial_y B_z \\ \partial_z B_z \end{array} \right)$$

Where $W_\perp = mv_\perp^2/2$. Immediately we find it produces a constant acceleration parallel to $B$. Let’s try to write this expression using general vector notation:

$$(\partial_x B_z, \partial_y B_z, \partial_z B_z) = \nabla B_z = \nabla (Bb)$$

$$= \nabla (B \frac{B}{B}) = \nabla B$$

Where $b$ is a unit vector parallel to $B$. Thus

$$F_{avg} = -\frac{W_\perp}{B} \frac{\nabla B}{B}$$

Inserting $F_{avg}$ into 2.8 gives the drift velocity

$$v_{\nabla B} = \frac{W_\perp B \times \nabla B}{q B^3}$$

This is called the gradient $B$ drift [3]. We notice that it depends on $q$, giving opposite values for positively and negatively charged particles and produces a net current and charge separation in a plasma. In addition to its dependence on $W_\perp$, the magnitude of the gradient drift falls as $1/B$, meaning it is smaller in stronger magnetic fields. Typical values inside a tokamak at fusion temperatures are on the order of 400 m/s, while electrons with energies of 0.1 MeV in the outer Van-Allen radiation belt have a drift speed on the order of 3 km/s.

3.2 Curvature drift

However, this result is incomplete. We assumed the guiding center velocity is parallel to $B$, even though the direction of $B$ changes as the particle moves. We will deal with this by viewing the
particle in an accelerated frame that keeps the direction of $\mathbf{B}$ at the particle’s position constant. This introduces a systemic centrifugal force $\mathbf{F}_{cf}$, while keeping our current equations of motion accurate. We can write

$$\mathbf{v}_{drift} = \frac{\mathbf{F}_{cf} \times \mathbf{B}}{qB^2} + \frac{W \mathbf{B} \times \nabla B}{qB^3}$$

All that is left now is to express $\mathbf{F}_{cf}$ as a function of $\mathbf{v}$ and $\mathbf{B}$. Since we are following the direction of $\mathbf{B}$ with speed $\mathbf{v}_\parallel$, the particle is affected by a centrifugal force

$$\mathbf{F}_{cf} = m\frac{\mathbf{v}_\parallel^2}{R_c} \mathbf{R}_c$$

where $\mathbf{R}_c$ is the local curvature radius vector of $\mathbf{B}$.

![Figure 4: Illustration of the curvature vector $\mathbf{R}_c$][2]

We express $\mathbf{R}_c$ with $\mathbf{B}$ using $d\mathbf{b}/ds = -\mathbf{R}_c/R_c^2$, where $s$ measures length along the field line, so that $\mathbf{b} = \mathbf{B}/B$. Since $d/ds$ is the derivative along the $\mathbf{b}$ direction, we can write

$$\frac{\mathbf{R}_c}{R_c^2} = (\mathbf{b}\nabla)\mathbf{b}$$

Giving

$$\mathbf{F}_{cf} = -mv\mathbf{v}_\parallel^2(\mathbf{b}\nabla)\mathbf{b}$$

Thus the drift equals

$$\mathbf{v}_{curv} = \frac{2W \mathbf{B} \times [(\mathbf{b}\nabla)\mathbf{b}]}{B^2}$$

This is known as the curvature drift [2]. Note that since $\mathbf{F}_{cf} \parallel \mathbf{R}_c \perp \mathbf{B}$, the only effect of this force is a drift, with no acceleration parallel to $\mathbf{B}$. Also note that once again the drift velocity depends upon $q$, and will thus drive a current and produce charge separation. Typical magnitudes for a fusion plasma in a tokamak are on the order of 400 m/s, and 800 m/s for electrons in the outer Van-Allen radiation belt.

Finally we can write the total guiding center drift as

$$\mathbf{v}_{drift} = \mathbf{v}_{\nabla B} + \mathbf{v}_{curv} + \mathbf{v}_{F} = \frac{W \mathbf{B} \times \nabla B}{qB^3} + \frac{2W \mathbf{B} \times [(\mathbf{b}\nabla)\mathbf{b}]}{qB^2} + \frac{\mathbf{F} \times \mathbf{B}}{qB^2}$$

And the guiding center acceleration as

$$\dot{\mathbf{v}}_{gc} = -W \mathbf{B} \nabla B + \frac{1}{m} \mathbf{F} = \dot{\mathbf{v}}_\parallel$$
Where $\nabla_B B$ denotes the rate of change of $B$ along field lines, that is, $\nabla_B B = \mathbf{b}((\mathbf{b} \nabla) B)$, and any force caused by $\mathbf{E}$ is included in $\mathbf{F}$. Note that the acceleration is parallel to $\mathbf{B}$.

We encounter such drifts when working with tokamaks - toroidal fusion devices. A tokamak is, in its most basic form, a simple magnetic coil shaped into a torus, creating a toroidal magnetic field. This is intended to trap charged particles without the losses that occur in a magnetic mirror (covered later). In a toroidal coil $B$ is larger closer to the inner wall, which means both $\nabla B$ and the curvature vector $\mathbf{R}_c$ point towards the inner wall, creating a vertical current due to the $\nabla B$ and curvature drifts, until the $\mathbf{E}$ field caused by charge separation balances the two drifts. However, this field gives rise to an $\mathbf{E} \times \mathbf{B}$ drift that drives the plasma towards the outer wall.

We can avoid this by adding a poloidal $B$ field, either by driving a current through the plasma, as is the case with tokamaks, or with additional or more complex coils, as is the case with stellarators. This causes particles to follow a helical path, mixing those on the top and those on the bottom, and thus undoing any charge separation that might occur.

Figure 5: $\nabla B$ and curvature drifts inside a tokamak. [3]

Figure 6: Combination of poloidal and toroidal $\mathbf{B}$ field in a tokamak. [3]
3.3 Magnetic mirror

We will show that particles can be reflected when traveling into a region with higher $B$. First we find the magnetic moment caused by our particle, since its gyration forms a current loop

$$\mu = \frac{mv^2_\perp}{2B}$$

Let us assume that $\mathbf{F} = 0$, $\mathbf{E} = 0$, so that the only field acting upon our particle is $\mathbf{B}$. Writing the guiding center acceleration, we get

$$m \frac{d}{dt} \mathbf{v}_\parallel = -W_\perp \frac{\nabla B}{B} = -\frac{mv^2_\perp}{2B} \nabla B$$

Multiplying both sides by $b$ yields

$$m \frac{dv_\parallel}{dt} = -\mu \frac{\partial B}{\partial s}$$

Where $B(s)$ runs parallel to $\mathbf{B}$. Since $\frac{ds}{dt} = v_\parallel$, we may multiply the left side by $v_\parallel$ and the right by $\frac{ds}{dt}$.

$$mv_\parallel \frac{dv_\parallel}{dt} = \frac{d}{dt} \left( \frac{1}{2} mv^2_\parallel \right) = -\mu \frac{\partial B}{\partial s} \frac{ds}{dt} = -\mu \frac{dB}{dt}$$

(3.4)

$\frac{dB}{dt}$ is the variation of $B$ as seen by the particle, while $B$ itself is not time-dependent. Since the only field is $\mathbf{B}$, the energy of the particle must be conserved

$$\frac{d}{dt} \left( \frac{1}{2} mv^2_\parallel + \frac{1}{2} mv^2_\perp \right) = -\frac{d}{dt} \left( \frac{1}{2} mv^2_\parallel + \mu B \right)$$

Where we used the relation $\frac{1}{2} mv^2_\perp = \mu B$. Combining this with eq. 3.4 yields

$$-\mu \frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0$$

Which finally gives

$$\frac{d\mu}{dt} = 0$$

(3.5)

Note that this only holds if $\mathbf{B}$, as seen by the particle, changes slowly compared to $\omega_c$. This will serve as the basis for constructing a magnetic mirror. We can arrange two magnetic coils symmetrically, so that they form a field $B_0$ in their center, and a stronger field $B_{max}$ nearer each coil. Suppose a particle is in the central $B_0$ region with $v_\parallel = v_\parallel 0$ and $v_\perp = v_\perp 0$. It will be reflected when $v_\parallel = 0$, $v_\perp = v_\perp'$ and $B = B'$. The invariance of $\mu$ yields

$$\mu_0 = \frac{mv^2_\perp 0}{2B_0} = \mu' = \frac{mv^2_\perp}{2B'}$$

And the conservation of energy yields

$$v'^2_\perp = v^2_\perp 0 + v^2_\parallel 0 = v^2_0$$

Combining the above equations gives

$$\frac{B_0}{B'} = \frac{v'^2_\perp}{v^2_\perp} = \frac{v^2_\perp 0}{v^2_0} = \sin^2 \theta$$

$$B' = \frac{B_0}{\sin^2 \theta}$$

Where $\theta$ is the angle between $\mathbf{B}$ and $\mathbf{v}$ in the weak-field region. This means that when the field reaches $B'$, a particle will be reflected, and if $B' > B_{max}$, a particle will not be reflected at all. This is the source of particle losses in a magnetic bottle made from two magnetic mirrors [2].
While not obvious at first glance, magnetic mirrors can appear in tokamaks as well. Since the magnetic field lines, augmented by the poloidal field, now follow a helical trajectory in- and out-of the inner, high-field side, a particle may bounce when it encounters the stronger $B$ field. The path such a particle follows is called a *banana orbit* [3].

The Van-Allen radiation belts, containing charged particles trapped in the Earth’s magnetosphere, are also a consequence of the mirror effect. As particles follow the field lines from the equatorial region towards the polar regions, they encounter a stronger magnetic field. Those that are reflected form the radiation belts, while those that are not produce the Aurora Borealis when they encounter the atmosphere [2].

4 Time-varying E field

4.1 Polarization drift

We take $E$ and $B$ to be spatially uniform, where $B$ is constant, while $E$ varies in time with a frequency much slower than $\omega_c$, and observe the effects such a setup gives. We recall the formula
for $\mathbf{E} \times \mathbf{B}$ drift

$$\mathbf{v}_{E \times B} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

Since $\mathbf{E}$ now varies with time, so does $\mathbf{v}_{E \times B}$

$$\dot{\mathbf{v}}_{E \times B} = \frac{d}{dt} \left( \frac{\mathbf{E} \times \mathbf{B}}{B^2} \right)$$

In the guiding center frame this is equivalent to a force $\mathbf{F} = -m \dot{\mathbf{v}}_{E \times B}$, which produces yet another drift

$$\mathbf{v}_P = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} = -\frac{m}{qB^4} \frac{d}{dt} \left( \mathbf{E} \times \mathbf{B} \right) \times \mathbf{B}$$

$$= -\frac{m}{qB^4} \frac{d}{dt} \left( (\mathbf{EB})\mathbf{B} - B^2 \mathbf{E} \right)$$

$$= \frac{m}{qB^2} \frac{d}{dt} \left( \mathbf{E} - (\mathbf{E} \times \mathbf{B})\mathbf{B} \right) = \frac{m}{qB^2} \mathbf{E}_\perp$$

This is called the polarization drift, and is dependent upon charge, and thus in opposite directions for positively and negatively charged particles, which produces a polarization current [4].

4.2 Cyclotron resonance

We shall now look at the effects of an $\mathbf{E}$ field that varies with the cyclotron frequency. We set $\mathbf{B} = (0, 0, B)$ and $\mathbf{E} = (E \cos(\omega_c t), 0, 0)$. The equations of motion become

$$\dot{v}_x = \omega_c v_y + qE \cos(\omega_c t)$$
$$\dot{v}_y = -\omega_c v_x$$
$$\dot{v}_z = 0$$
$$\ddot{v}_x = \omega_c \dot{v}_y - \omega_c qE \sin(\omega_c t) = -\omega_c^2 v_x - \omega_c qE \sin(\omega_c t)$$
$$\ddot{v}_y = -\omega_c \dot{v}_x = -\omega_c^2 v_y - \omega_c qE \cos(\omega_c t)$$

Let's guess that the solution will be of the form

$$v_x = A \cos(\omega_c t)$$
$$v_y = A \cos(\omega_c t + \delta)$$

(4.4)

(4.5)

Inserting 4.4 and 4.5 into 4.1 gives $\delta = \pi/2$, while inserting 4.4 into 4.2 yields

$$-2A\omega_c \sin(\omega_c t) - A\omega_c^2 \cos(\omega_c t) = -A\omega_c^2 \cos(\omega_c t) - E\omega_c q \sin(\omega_c t)$$
From which we can derive $A$ to be

$$A = \frac{1}{2}Eq$$

Thus the solution is

$$v_x = \frac{1}{2}Eq t \cos(\omega_c t)$$
$$v_y = -\frac{1}{2}Eq t \sin(\omega_c t)$$

We can express $v_\perp$ and $W_\perp$ as

$$v_\perp = \sqrt{v_x^2 + v_y^2} = \frac{1}{2}Eq t$$
$$W_\perp = \frac{1}{2}mv_\perp^2 = \frac{1}{8}mE^2q^2t^2$$

Figure 10: Spiral trajectory of a particle undergoing cyclotron heating.

We see that $W_\perp$ increases as $t^2$! This is the basis for *cyclotron resonance heating* - plasma is irradiated with radio waves of frequency $\omega_c$, thus heating it. Since ions and electrons have different $\omega_c$, we can choose which of them to heat directly. Furthermore, since $B$ decreases with distance from the inner wall of a tokamak, so does $\omega_c$, meaning we can control where the energy of cyclotron heating is deposited by choosing an appropriate frequency. [3]

5 Conclusion

We have shown that charged particles follow complex paths in an EM-field, and some of the effects that make containing charged particles difficult. However, missing from our analysis are multi-particle effects, such as collisions with other particles, and collective behavior - the interaction of a plasma with its own EM-fields, which further complicates the containment of a hot plasma. Nonetheless such simple theory can often produce workable results, especially if the particle density is low enough to be able to neglect collision effects, and the EM-field caused by the plasma is small compared to the external field, as is the case in some space plasmas.
References


