Univerza v Ljubljani
FMF - Fakulteta za matematiko in fiziko
Course: Seminar 2

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The stability of the bicycle

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Summary

We review the stability of the bicycles. We show that a bicycle is almost, though not quite, self-stable; the most important parameter governing the stability is the “castor” of the front wheel. The other contributor to stability is gyroscopic effect. The almost stable behavior explains why a bicycle is so easy to ride at speed.

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Introduction

Why a bicycle rider does not fall over?
Almost everyone can ride a bicycle, yet no one asks why. It turns out the question is not so trivial. A simple answer would be something like: the bicycle is assumed to be balanced by the rider who, when feeling the vehicle falling, steers into direction of the fall and so traverses a curved trajectory of such radius as to generate appropriate centrifugal force to avoid the fall. It turns out that it is not as trivial as it appears at first glance.

History

Timoshenko and Young in 1948 [1] explained from physical point of view many every-day tools and resources. Among others, they precisely described the motion of the rolling coin and suggested something similar to be valid for bicycle. They derived the solution of equations of motion with conservation of angular momentum thereby suggesting the gyroscopic effect to be the cause of the bicycle stability. Since the simple model of bicycle would need some sort of a tail attached to the coin their description was not entirely valid or not valid at all.

After Timoshenko and Young, Jones in 1970 [2] published an article about the problem. He began the discussion of bicycle stability with creation of "unridable bicycles''. He created a non-gyroscopic bicycle by mounting an extra wheel alongside the front wheel, but a few centimeters above the ground so it could be spun in the opposite direction, thereby canceling the angular momentum of the front wheel (Fig. 1).

Figure 1. Jones's non-gyroscopic bicycle [2].

He found this bike easy to ride. However, without a rider this kind of bicycle fell over much sooner. Jones concluded that the light riderless bicycle is stabilized by gyroscopic action, whereas the heavier ridden model is not. It let him to seek an alternative theory.
Jones soon suspected the steering geometry to play a crucial role. He found out the bicycle could achieve a lower gravitational potential energy by steering the front wheel if leaned. For every angle of lean there is a different steering angle (Fig. 2) providing lowest potential energy of the bicycle (Fig. 3).

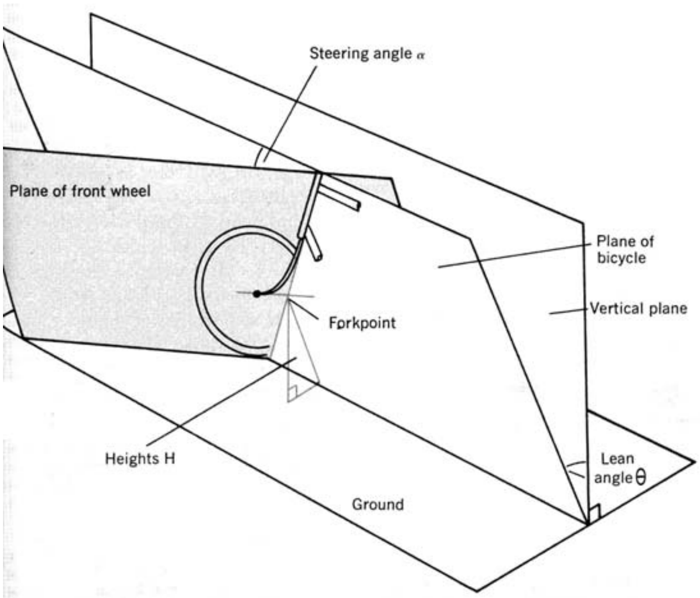

Figure 2. Sketch of leaning bicycle and definition of the steering angle *α* and the angle of leand *θ* [2].

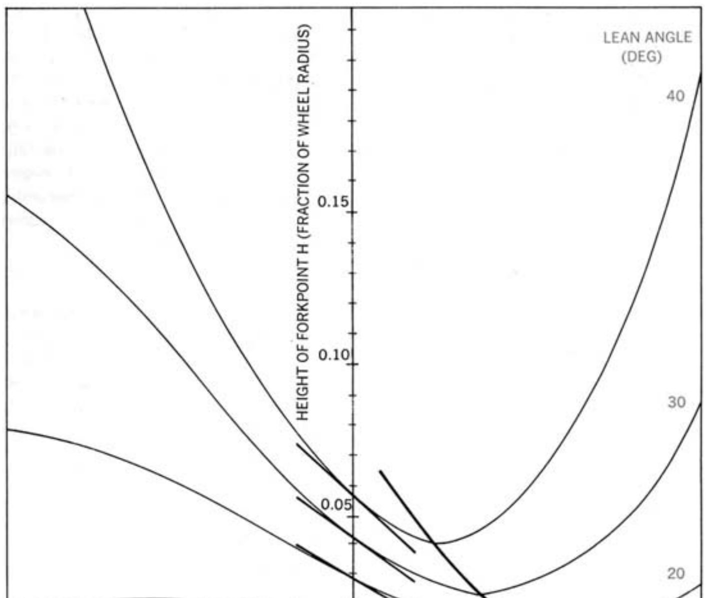

Figure 3. Plot of steering angle versus height of forks for different lean angle. Minimal height or the lowest potential energy occurs closer to straight-ahead position for higher values of lean angle. The tich line connects the minima at different lean angle [2].

He considered $\frac{d^2H}{d\alpha \cdot d\theta}$ where H is the height of the forks, α is the angle of the front wheel, and θ is the angle of the leaning plane or lean of the bicycle (Fig. 3), to be a parameter of bicycle stability.

This parameter is an effect of the trail or castor, that is, the distance from the front wheel contact point to the point where the steering axis intersects the ground (Fig. 4).

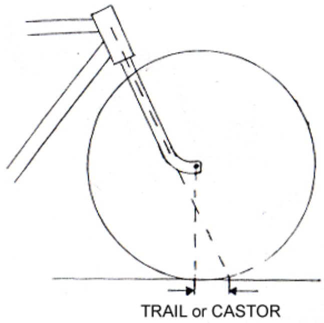

Figure 4. Definition of the trail or castor [3].

He supported his theory by showing all bicycles had a value of $\frac{d^2H}{d\alpha \cdot d\theta}$ in the "stable region" which means that the value should be positive. That is, for stability the height of the forks falls as the wheel turns into the lean when the bike is tilted. He also constructed a bicycle (Fig. 5) with a negative $\frac{d^2H}{d\alpha \cdot d\theta}$ which was in "unstable region" and therefore to be hard to ride.

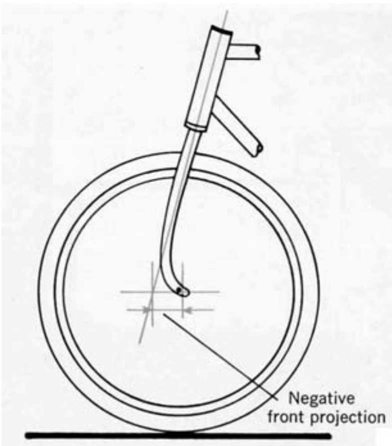

Figure 5. Bicycle with negative castor [2]

He concluded that for stability of the bicycle the steering system is much more important than the gyroscopic effect.

Eight years later Kirshner [3] published an article revealing the parameter $\frac{d^2H}{d\alpha \cdot d\theta}$ to be approximately the castor or trail T (Fig. 6).

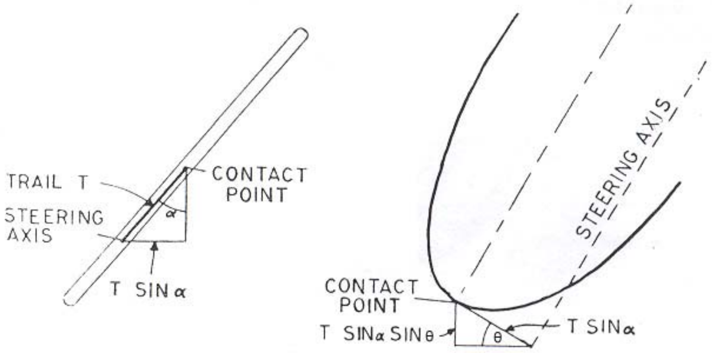

Figure 6. Top view of the front wheel (left). As the bicycle is steered, the front wheel rotates about the contact point, moving the bicycle sideways a distance T sin *α*, where T is the trail (left). Front view of front wheel (right). For a leaning bicycle this creates a vertical displacement of H = T sin *α* sin *θ*. The frame of the bicycle is perpendicular to the plane of the paper [3].

In Fig. 5 we see that for small angles α and θ the vertical siplacement of the center of mass H is $T\sin\alpha\sin\theta$, or approximately $T\alpha\theta$. It is obvious that $\frac{d^2H}{d\alpha\cdot d\theta}$ is T .
Kirshner wondered if it is possible to make a quasistatic bicycle which would ride itself in a circular path with a constant velocity, constant angle of steer and constant angle of lean. He analytically constructed the "effective potential'' which included centrifugal and gravitational potential. The result was that at higher velocities the minimum of potential energy curve, that is, the equilibrium steering angle, occurs closer to the straight ahead position which could explain quasistatic bicycle. But after many simulations and experiments it was clear that the bicycle always falls over regardless of the velocity and the mass of the bicycle (with or without the rider). The quasistatic bicycle is impossible to construct. The conclusion was the steering geometry probably is an important factor in bicycle stability, but does not in itself explain the self-stability. Therefore front wheel, acting as a gyroscope, makes an important contribution to balance a bicycle.

The mechanism of bicycle stability was revealed in the article by Lowell and McKell from 1982 [4]. They found that castor and gyroscopic effect work hand in hand to provide the necessary stability.

Defining our model of bicycle

We begin our discussion with a simplified bicycle frame model (Fig. 7).

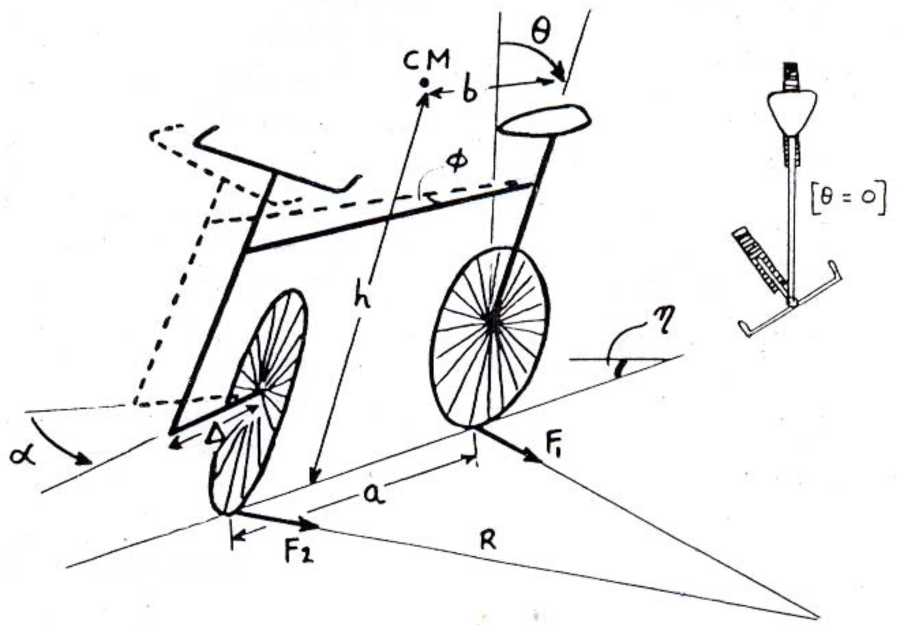

Figure 7. Sketch of simplified bicycle showing the parameters important in the analysis of bicycle stability. CM is the center of mass of the bicycle with rider. *R* is the radius of the bicycle's path [4].

The acceleration of the center of mass consists of three terms. The first one $b\frac{d^2(\sin\eta)}{dt^2}$ is the acceleration caused by turning the center of mass around axis at the hub of the rear wheel which is perpendicular to the ground. The second term $h\frac{d^2(\sin\theta)}{dt^2}$ is the acceleration caused by turning the center of mass around contact points of wheels and ground. The third term $\frac{v^2}{R}$ is caused

by the center of mass turning around axis which is perpendicular to the plane of the path of the bicycle and intersects that plane at equidistant point to wheels.
If the angles θ and η are small, the acceleration of the center of mass (CM in Fig. 6) in the direction normal to the frame is

$$b\ddot{\eta} + h\ddot{\theta} + \frac{v^2}{R}, \tag{1}$$

where R is the radius of the path of the bicycle and v is the speed of the bicycle; h and b are defined in Fig. 6. We see that

$$R\dot{\eta} = v \tag{2}$$

and

$$R\alpha = a. \tag{3}$$

The component of the weight normal to the frame is

$$Mg\sin\theta \approx Mg\theta. \tag{4}$$

If we insert Eqs. (2-4) into Eq. (1), we derive the equation

$$\ddot{\theta} + \frac{bv}{ha}\dot{\alpha} + \frac{v^2}{ha}\alpha - \frac{g}{h}\theta = 0 \tag{5}$$

which describes connection between angle of lean θ and steering angle α. A change in θ automatically causes a change in α which through Eq. (5) tends to restore θ to its original value.

Stabilization mechanisms

When the bicycle leans over which means that θ increases, the angular momentum of the front wheel must imply that it precesses, which means that α increases. Let us define I_z as the moment of inertia of the front wheel about the steering axis and I as moment of inertia about its axis of rotation. If we assume there is no friction turning the front wheel and no torque on the handlebars, we can use conservation of angular momentum about the steering axis.

$$\frac{d}{dt}(I_z\dot{\alpha} - I\omega\sin\theta) = 0 \tag{6}$$

and for small θ

$$\ddot{\alpha} = \frac{Iv}{I_z r}\dot{\theta} \tag{7}$$

where ω is the angular speed of the front wheel and r is the radius of the wheel. It is obvious that the front wheel turns into the same direction as bicycle is falling. So it arrests the fall.

Evidently, the Jones's castor too is important. Turning the handlebars through an angle α will cause the crossbar to turn through angle $\phi \approx \frac{\alpha\Delta}{a}$ (Fig. 7), and the center of mass to move a distance $b\phi$ normal to the frame. If the bicycle is leaning at an angle θ, the center of mass will fall, when the handlebars turn through a distance $b\phi\sin\theta \approx \frac{b\Delta}{a}\alpha\theta$. Turning the handlebars through α therefore lowers the potential energy by $\frac{Mgb\Delta}{a}\alpha\theta$, where M is the mass of the cycle with rider. There is therefore a torque on the front wheel whose magnitude is $\frac{Mgb\Delta}{a}\theta$ and whose direction tend to increase α. Thus

$$I_z\ddot{\alpha} = -\frac{Mgb\Delta}{a}\theta\,. \tag{8}$$

Again, the wheel turns into the fall, which tends to reduce θ.

The castor Δ also causes a torque on the front wheel which tends to reduce α. This torque is $F_2\Delta$, where F_2 is the sideways frictional force on the front wheel. The frictional forces F_1 and F_2 together provide the force necessary to maintain the bicycle on its circular path; therefore $F_1 + F_2 = \frac{Mv^2}{R}$. To prevent rotation about vertical axis through the center of mass, $\frac{F_2}{F_1}$ must be equal to $\frac{b}{a-b}$. Thus

$$I_z\ddot{\alpha} = -\frac{Mb\Delta v^2}{a^2}\alpha\,. \tag{9}$$

Now collecting the contributions to $\ddot{\alpha}$ [Eqs. (7-9)] gives

$$\ddot{\alpha} - \mu\left(\theta - \frac{v^2}{ga}\alpha\right) - \chi v\dot{\theta} = 0 \tag{10}$$

where $\mu = \frac{Mbg\Delta}{I_z a}$ and $\chi = \frac{I}{I_z r}$.

Now we have two equations Eq. (5) and Eq. (10) describing how the bicycle reacts to a sudden change of angle of lean θ with changing the steering angle α.

To examine stability, we assume that the bicycle is initially in equilibrium and upright $(\theta = \ddot{\alpha} = \dot{\theta} = \dot{\alpha} = \ddot{\theta} = \ddot{\varepsilon} = 0)$ and study the effect of an impulse which imparts an initial rate of fall $(\dot{\theta} = W_0)$.

Self-stability without gyroscopic effect

Absence of gyroscopic effect would require $\chi \to 0$. If we measure the magnitude of μ we see it to be rather large, therefore the $\theta - (v^2/ga)\alpha$ must be small because if it were much greater than 0 $\ddot{\alpha}$ would be large and α would rapidly grow until $\frac{v^2}{ga}\alpha$ became almost equal to θ. Therefore we try with an approximation

$$\theta \approx \frac{v^2}{ga}\alpha\ . \tag{11}$$

Substituting for α in Eq. (5) gives

$$\ddot{\theta} + \frac{gb}{hv}\dot{\theta} = 0 \quad . \tag{12}$$

The solution to equation is

$$\theta = \frac{gb}{hv} W_0 \left(1 - e^{-\frac{gb}{hv}t}\right) \tag{13}$$

So if for some reason the bicycle starts falling over at $t = 0$, the fall does not continue but instead, θ asymptotically approaches to value of $hv\frac{W_0}{gb}$, and α approaches $ha\frac{W_0}{gb}$. This means that the bicycle falls into a circular path of radius $W_0\frac{bv}{h}$. The numerical solution of Eq. (5) and Eq. (10) can be exactly obtained. If we plot the exact solutions of angle θ and angle α versus time after impulse we see additional oscillations (Fig. 8).

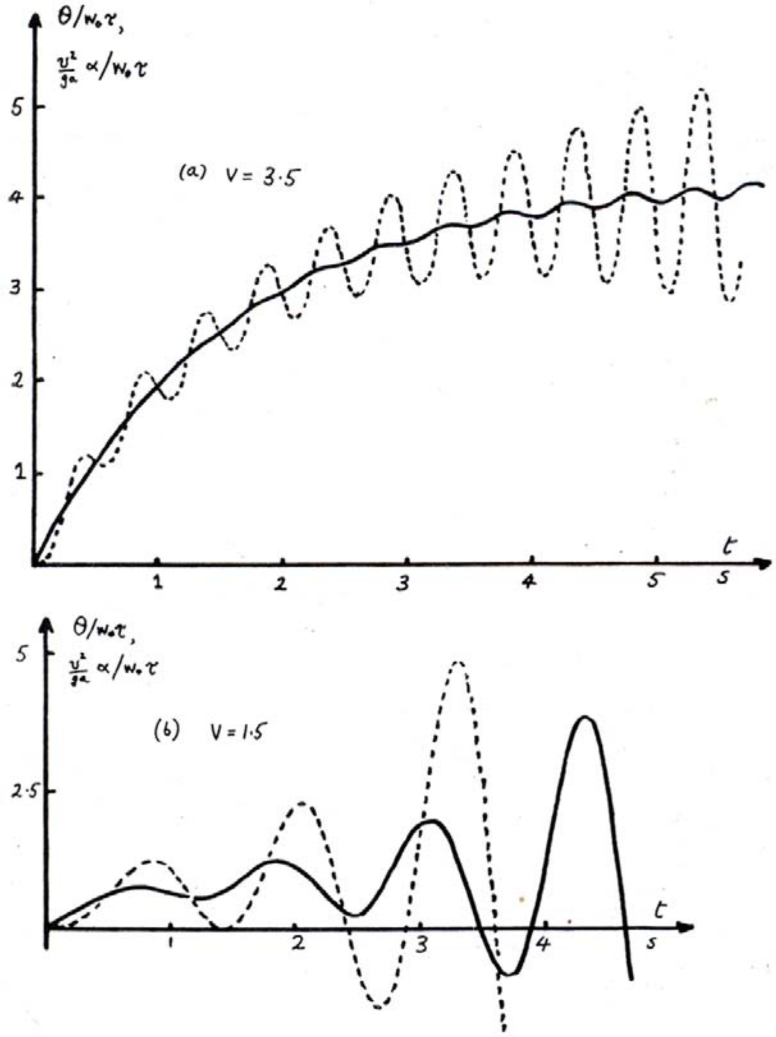

Figure 8. Simulated numerical solution of the response of a bicycle to an impulse tending to fall over. Solid curve: angle of lean (θ) versus time; dotted curve: angle of front wheel (α) versus time. Both θ and α are measured in units of $W_0\tau$, where *Wo* is the initial rate of change of θ and τ is the characteristic time $(h/g)^{1/2}$. Curves (a) are for speed of 3.5 m/s and (b) for 1.5 m/s. All bicycle parameters needed to solve the equation had been measured on real bicycle [4].

To get an intuitive feeling of the origin of the oscillations we can proceed as follows. If we put $\chi \to 0$ into Eq. (10) we obtain

$$\ddot{\alpha} + \frac{\mu v^2}{ga}\alpha = \mu\theta . \qquad (14)$$

If for some reason α deviates from zero and θ were constant we get a harmonic oscillator. But because θ is not constant, in fact oscillations in α causes oscillation in θ, we get a shifted phase. In fact we get a driven harmonic oscillator with growing α.

The bicycle is almost self-stable but let us see what is the magnitude of the contribution of the gyroscopic effect.

Self-stability with gyroscopic effect

If $\chi \neq 0$ and the gyroscopic term $\chi v \dot{\theta}$ is included in Eq. (10), we can take a look at the full numerical solution (Fig. 9).

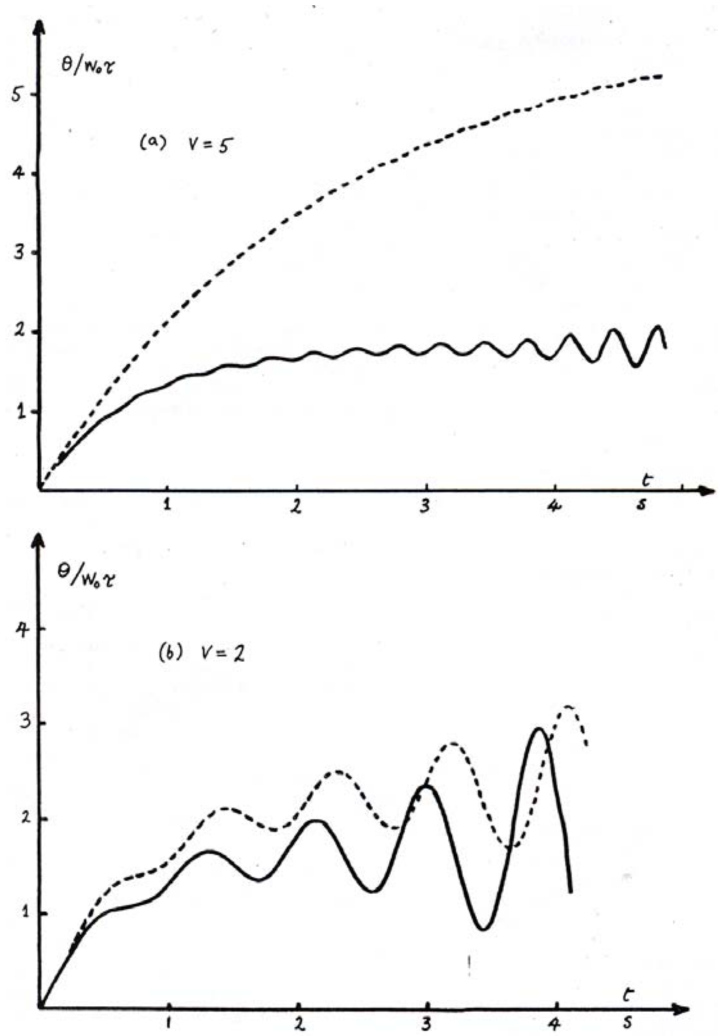

Figure 9. Influence of gyroscopic effect of the front wheel on bicycle stability. The dotted curves is a simulation without gyroscopic effect and the solid curves is a simulation with gyroscopic effect. The gyroscopic effect is stabilizing in the sense that result in a smaller mean value of *θ*, but destabilizing in the sense that it enhances the oscillatory instability. Curve (a) speed 5 m/s; (b) 2 m/s [4].

We see now that the gyroscopic effect reduces the mean value of θ, thereby increasing stability. However, the same effect enhances the oscillatory instability. The second effect prevails at higher speeds.

We see that $\chi = \dfrac{I}{I_z r}$ is dependent of the radius of the wheel. If r were increased, χ would decrease, χ, therefore the bicycle would become less gyroscopic.

Now we also have an answer why μ is usually in real bicycles although the solution of Eq. (12) is stable at much lower values $\mu \geq 15$. It prevents the growth of instability caused by the gyroscopic effect (Fig. 10). This is the answer to importance of Jones's trail and his unrideable bicycles [2].

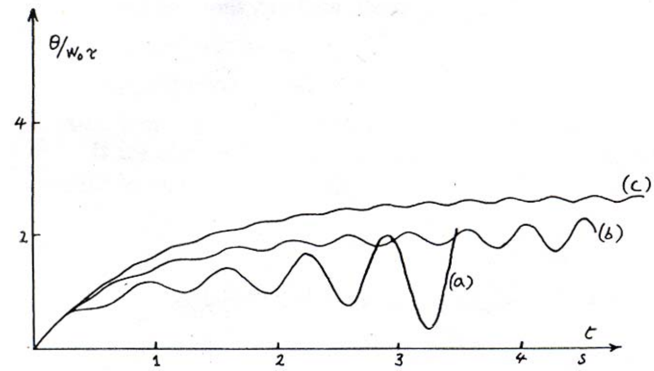

Figure 10. Solutions for leaning angle with different castors as measured by the parameter *μ*. Curve (b) is the predicted response of a real bicycle. Curve (a) shows the predicted response if the castor were halved ($\mu = 66.5\ s^{-2}$), and curve (c) is the prediction for twice the castor ($\mu = 266\ s^{-2}$). Speed is 5 m/s [4].

Conclusion

The most important factor in self-stability of the bicycle is the castor or trail of the front wheel, as suggested by the experiments of Jones [2]. However the gyroscopic effect decreases the growth of the leaning angle when the bicycle is disturbed. Therefore it contributes to the self-stability. In a sense the rider has more time to act and straighten the bicycle.

The model of the bicycle discussed in this paper is not as accurate as it should be in order to precisely explain the stability of a real bicycle. We assumed the thickness of the front tire is zero, the front forces is perpendicular to the ground whereas in case of a real bicycle it is not. Furthermore the rider moves sideways when paddling, moving the center of mass sideways with some frequency which also imposes some oscillation to the motion. But these are small contributions therefore our model of bicycle works fine for many cases.

It is important to mentions that many aspects should be considered when designing a bicycle for certain use and the stability is among less important. Among others there are strength, size, weight, etc. For example the castor in a case of a child’s bicycle is small because of the size of the frame and castor in case of down-hill bicycle is large due to required control when riding on steep terrain. Thereby there are many different bicycles with different castors. To sum up the castor by itself effects on the stability of the bicycle but is primarily result of other required performances of the bicycle.

The motorbikes are stabilized in a same way as bicycle; the only difference is that the castor is made to provide self-stability at higher speeds and larger combined mass of the motorbike and a rider. The interesting exceptions are the special designed choppers. From esthetic reasons they have extended forks which results in a higher value of castor. This kind of motorbike lacks self-stability and is hard to balance.

It is interesting that the stability of the bicycle was not understood earlier. After all, almost everyone learns to ride a bicycle at one point in his life. On the other hand, we see that there are many things surrounding us worth asking ourselves: how does it work. Even a vehicle as simple as a bicycle is full of interesting physical phenomena.

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