Options pricing in discrete systems

Seminar II

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Abstract

This paper is a basic introduction of options and application of mathematics in calculating their values. At first an extended introduction of options properties is given as a motivation, as well as for easier understanding of options pricing. In the main part we consider the discrete-time binomial model for calculating the price, and main assumptions made for deriving it. We end with a taste of stochastic calculus, which is needed for a more advanced level of option pricing theory.

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1. Introduction
The reason for this seminar is to present the possibility for application of mathematics and physics in finance. Many times processes in finance are modeled with processes in physics and use the same mathematics. Experience from physics is therefore very useful in understanding financial principles and models. Options are only one of the instruments in finance, but may be one of the most interesting from the viewing point of a physicist.

The strongest link that bonds mathematics with the stock market is the application of stochastic calculus, which is in many ways similar to application in physics. Though this is best seen in continuous time version of options pricing, we will explore the more basic and easy to understand discrete model. Continuous model follows from the same principles and is an upgrade for which a Nobel Prize in finance was awarded. The so called Black-Scholes PDE, which is the core of the continuous model, is a success story of mathematics in finance and has been a matter of study and modification ever since its emersion.

In the following chapters we will get to know the properties of options, the reason for their existence and their application. Most importantly we will meet the basic principles used in options pricing, derive its conclusions and finish off with a more sophisticated derivation of value of a specific type of option, just to get a taste of stochastic calculus.

2. What are options
One or another type of options has been around since the middle ages. But it was not until 1973 that option standardization followed the standardization of futures with the establishment of Chicago Board of Options Exchange (CBOE), which is the largest options exchange to date. Before, options were sold each day separately at the same maturity (i.e. 3 months) with the exercise price being the same as the price of the commodity (usually options were written on commodities) thus preventing the formations of an exchange market. From 1973 options are sold with fixed expiry dates every three months and the exercised prices are standardized in bands i.e. $39, $40 and $41 for a stock currently priced at $40.

To understand how pricing of an option works, we must first understand its features. Why they are so popular amongst traders when it comes to cutting risk and why one should not take their use lightly. First we start off with definitions of terms that are perhaps unfamiliar. Later in the chapter we work our way up to specifying different option types, discussing their unique features which helps us explain their common use.
2.1 Important terms and concepts

2.1.1 Option

DEFINITION: An option is a contract which confers the right but not the obligation to buy or to sell an asset on a given date at a predetermined (exercise) price.

The most important thing about options is that the option buyer has the right to abandon the contract, if it is not profitable. He only risks losing the price of the option bought. This causes an asymmetric payoff for the buyer and the seller of an option, as it will be discussed later on.

2.1.2 Exercise and related terms

- **Exercise** in this case denotes the going through with the deal;
- **exercise price (strike price)** is the predetermined price at which the deal goes through;
- **expiry date** is the preset date on which the option may be exercised;
- **early exercise** is the exercise before the expiry date which is possible with some types of options;
- **option maturity** is the time left until the expiry date

2.1.3 Other terms

- A **forward contract** is an agreement between two parties to buy or sell an asset (which can be of any kind) at a pre-agreed future point in time
- A **future contract** is a standardized contract, traded on a futures exchange, to buy or sell a certain underlying instrument at a certain date in the future, at a specified price
- **Hedging** is a strategy designed to minimize exposure to an unwanted business risk, while still allowing the business to profit from an investment activity
- **Volatility** most frequently refers to the standard deviation of the change in value of a financial instrument within a specific time horizon
- **Risk-free interest rate** is the interest rate obtained by investing in financial instruments with no default risk (like bonds).
- A **lookback option** is a path dependent option where the option owner has the right to buy (sell) the underlying instrument at its lowest (highest) price over some preceding period

2.2 Types of options

On a basic level there are two different types of options. We have the *call* option and very much the opposite is the *put* option.

Call options are far more common and their usage is easier to understand. They are options to buy assets and are usually bought by investors. A profit made by the call option is shown in the graph (Picture 1). The most important feature is that the loss is bounded whereas the profit is not. To put it in simple language, we surrender part of our profit in order to cut potential losses. If you are a call buyer, you are hoping that the share price will rise above the exercise price, otherwise the call is worthless and you lose the money spent on options. On the other hand the person that is selling the

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An option is counting on the price being below the exercise price, thus making a maximum profit, which is the revenue of calls sale. Contrary to the buyer, the seller's loss is possibly unbounded.

![Picture 1: profit made by a call option](http://en.wikipedia.org/wiki/Image:CallOption.png)

As mentioned before and one can guess on his own, put options are options to sell assets. This is usually the domain of producers of commodities that want to guarantee the sale of their product at the exercise price. Again the profit on put options for the buyer is unbounded, while the loss is limited with the put options price (Picture 2). The difference to a call option is that in this case the profit is made when the price of the share falls below the exercise price. If the prices stay above it, the trader will abandon put options. Again the effects on the put seller are mirrored from the buyers.

The secondary types are the so called *European and American* options. Both types can be applied to the previously described ones, making a family of 4 different options types. The difference between the European and American option is the exercise date. While the European option can only be exercised on the expiry date, the American option can be exercised any time prior to the date, including the date itself. The difference is important when evaluating the price of an option as we note that American options should be worth at least as much as European ones, for they give the owner greater flexibility.

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2.3 Use of options
Options are most commonly used in hedging strategies. Combining different option types and exercise values gives us risks and potential profit according to our desires. The idea of hedging with options is securing part of assets, while cutting potential regret for lost profit. It is of course more risky than bonds which pay the risk free interest rate, but more secure than stocks.

Strategies based on options can be divided into vertical, horizontal and diagonal spreads. Vertical spreads use options at different exercise prices but with same maturity, whereas horizontal spreads use options with the same exercise price, but different maturities. Diagonal spreads use both different exercise prices as well as different maturities.

Further exploration of strategies is not a matter of this seminar so we will skip it and continue with pricing of options.

3. Option pricing

3.1 Important terms and concepts
- Present value (discounted value) is the value on a given date of a future payment or series of future payments, discounted to reflect the time value of money and other factors such as investment risk
- Intrinsic value is the value upon immediate exercise
- Arbitrage is the practice of taking advantage of a price differential between two or more markets
- Derivatives are financial instruments whose value is derived from the value of something else
- Option replication is a term used for matching the payoff of an option with investments in stock (or another) market and money (risk-free) market.

3.2 Informal evaluation of call option value
Just to get started, let me show what option pricing is all about in a short example. Consider an investor in housing that wants to buy an apartment building. Currently he can get it for £ 100000 (S) and he can rent it for £ 4000 (D) a year. The other possibility is to acquire an option to buy an apartment in a year’s time for a price of C at an exercise price of £ 102000 (E). The important thing to consider when comparing the cases is that it is not the exercise price that is spent, but its present value. It means that you need to invest £ 92727 (PV(E)) now, if risk-free interest rate is 10%, to have £ 102000 in time to exercise the option.

Perhaps a comment on both strategies. They do both guarantee the apartment, but the strategy with options guarantees £ 102000 even if the market prices of housing falls. Therefore the call option strategy must be worth at least as much as the direct buying one. We obtain an inequality:

\[ C + PV(E) \geq S - D \]

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In this case:

\[ C \geq \£ 3273 \]

If the prices are volatile it might be worth much more than the lower bound. If we summarize, the influencing factors on call option price are (with the direction being denoted with (+) and (-)).

- Asset price (S) +
- Dividends or rents (D) –
- Time to maturity (t) +
- Exercise price (K) –
- Interest rate (r) +
- Volatility of asset price (\( \sigma \) ) +

This is a very informal evaluation and will be followed by more sophisticated means of pricing.

### 3.3 Binomial model

The center of attention in this seminar is the binomial evaluation model. This is a numerical model which divides continuous time into discrete periods in which the prices are calculated. It is based on probability and is best described by coin tossing, where the coin does not have to be fair. The idea is that heads means stock goes up by a factor \( u \), and tails means stock goes down by a factor \( d \) as shown in the picture (Picture 3). These two factors have to be chosen based on predictions of market volatility in the period before expiration. For creating a no-arbitrage pricing model we need the condition

\[ 0 < d < 1 + r < u, \]

where \( r \) is the risk-free interest rate. The down factor has to be positive as stock prices are, and it has to be smaller than growth from risk-free interest rate, otherwise any movement (up or down) gives us greater profit than the risk-free investment (arbitrage). On the other hand the up factor has to be larger than growth from risk-free interest rate, otherwise investment in stock wouldn’t make sense, as profits are always smaller than profits from risk-free investments. We usually also assume:

- **shares of stock can be subdivided for sale or purchase**
- **the interest rate for investing is the same as the interest rate for borrowing**
- **the purchase price of stock is the same as the selling price (zero bid-ask spread)**
- **at any one time the stock can take only two possible values in the next period.**
2.3.1 One period model

- One period example

The binomial model uses option replication with stock and money market investments to determine the value of an option. The easiest way to show this is with an example. Say we have \( S(0) = 4 \), \( u = 2 \), \( d = \frac{1}{2} \) and \( r = \frac{1}{4} \). Then \( S_1(H) = 8 \) and \( S_1(T) = 2 \). If we want to know the value of an option with the exercise price of 5 we can replicate it in a following way. Let’s say we have an initial wealth of \( X_0 = 1.20 \) and want to buy \( \Delta_0 = \frac{1}{2} \) shares of stock at time zero. Then we must borrow an additional 0.80 at an interest rate of \( r \). Our cash position at time zero is then \( X_0 - \Delta_0 S_0 = -0.80 \), and becomes \( (1 + r) (X_0 - \Delta_0 S_0) = -1 \) at time 1. But we will also have stock valued either \( \Delta_0 S_1(H) = 4 \) or \( \Delta_0 S_1(T) = 1 \), depending on the outcome of the coin toss on time 1. Calculating the final portfolio value we have

\[
X_1(H) = \Delta_0 S_1(H) + (1 + r) (X_0 - \Delta_0 S_0) = 3,
\]

and

\[
X_1(T) = \Delta_0 S_1(T) + (1 + r) (X_0 - \Delta_0 S_0) = 0.
\]

Thus we match the payoff of an option with an exercise price of \( K = 5 \). The conclusion is, that if the payoff is the same, its value at time 0 \( (V_0) \) must match the initial financial position, being 1.20. This is called the principle of arbitrage. If it is not met, a possibility of arbitrage is present.

Derivation of important quantities for option replication in one period model

The quantities we are interested in are the hedge ratio ($\Delta_0$), which is crucial for option replication, and the price of option at time zero ($V_0$). Using the experience from the previous example we have the equation:

$$X_1 = (1 + r) X_0 + \Delta_0 (S_1 - (1 + r) S_0).$$

We want to calculate $X_0$ and $\Delta_0$ so that $X_1(H) = V_1(H)$, and $X_1(T) = V_1(T)$ ($V_1$ being the value of an option at time one, which is essentially the payoff on exercise at time one). We have a system of two equations:

$$X_0 + \Delta_0 \left( \frac{1}{1 + r} S_1(H) - S_0 \right) = \frac{1}{1 + r} V_1(H),$$

$$X_0 + \Delta_0 \left( \frac{1}{1 + r} S_1(T) - S_0 \right) = \frac{1}{1 + r} V_1(T).$$

To solve these equations and extract $X_0$ and $\Delta_0$ we can multiply the first with a factor $\bar{p}$ and the second with $\bar{q} = 1 - \bar{p}$, for $\bar{p}$ chosen to comply with the equation:

$$S_0 = \frac{1}{1 + r} (\bar{p} S_1(H) + \bar{q} S_1(T)).$$

We will discuss the meaning of these factors in a little while, but for now this constraint is enough. We add the multiplied equations, consider the constraint, and we get:

$$X_0 = \frac{1}{1 + r} (\bar{p} V_1(H) + \bar{q} V_1(T)).$$

Directly from the constraint we also get, by substituting $S_1(H)$ with $uS_0$ and $S_1(t)$ with $dS_0$:

$$\bar{p} = \frac{(1 + r) - d}{u - d}, \quad \bar{q} = \frac{u - (1 + r)}{u - d}.$$

The final step of the calculation is subtracting the multiplied equations, which yields:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}.$$

Again we know, from previous experience, that we can price the option at time 0 as:

$$V_0 = \frac{1}{1 + r} (\bar{p} V_1(H) + \bar{q} V_1(T)).$$

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6 Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.6
In addition we can discuss the meaning of artificially produced factors $\tilde{p}$ and $\tilde{q}$. They can be considered as probabilities as they have all the features required. Exploring their meaning we come to a conclusion, that these are the probabilities of heads and tails respectively in a risk-neutral model, where the mean profit matches the risk-free interest rate. These are not actual probabilities, but constructs for calculating the option value. Actual probability for heads is usually strictly greater than the risk-neutral one ($\tilde{p}$), otherwise investing in the stock market would not make sense.

One period model is the base for the actual application of the binomial model – the multiperiod binomial model.

2.3.2 Multiperiod binomial model

This is an upgrade of the one-period model, meaning we have a series of coin tosses that determine the outcome. Usually numbers greater than 100 are used which yields $2^{100} \approx 10^{30}$ possible outcomes or more. So after deriving the recursive formulas for calculating option price in theory, we will have a look at practical implementation of multiperiod binomial model.

Switching to several coin tosses in a row, thus simulating change of stock price in time, asks for a generalization. For now we will be satisfied with European type of options, meaning no possibility of early exercise. The core of the multiperiod binomial model is the following theorem:

- **Theorem: Option replication in the multiperiod binomial model**

Consider an $N$-period binomial asset-pricing model, with $0 < d < 1 + r < u$, and with

$$\tilde{p} = \frac{1 + r - d}{u - d}, \quad \tilde{q} = \frac{u - (1 + r)}{u - d}.$$ 

Let $V_n$ be a random variable (a derivative security paying off at time $N$) depending on the first $N$ coin tosses $\omega_1\omega_2...\omega_N$. Define recursively backward in time the sequence of random variables $V_{N-1}, V_{N-2},..., V_0$ by

$$V_n(\omega_1 \omega_2 ... \omega_n) = \frac{1}{1 + r} \left[ \tilde{p} V_{n+1}(\omega_1 \omega_2 ... \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \omega_2 ... \omega_n T) \right] ,$$

so that each $V_n$ depends on the first $n$ coin tosses $\omega_1\omega_2...\omega_n$, where $n$ ranges between $N-1$ and $0$. Next define

$$\Delta_n(\omega_1 ... \omega_n) = \frac{V_{n+1}(\omega_1 ... \omega_n H) - V_{n+1}(\omega_1 ... \omega_n T)}{S_{n+1}(\omega_1 ... \omega_n H) - S_{n+1}(\omega_1 ... \omega_n T)} ,$$

where again $n$ ranges between $0$ and $N - 1$. If we set $X_0 = V_0$ and define recursively forward in time the portfolio values $X_1, X_2,..., X_N$ by the wealth equation

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7 Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.12 (Theorem 1.2.2)
\[ X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n), \]

then we will have

\[ X_n(\omega_1 \ldots \omega_N) = V_n(\omega_1 \ldots \omega_N). \]

\begin{itemize}
  \item \textbf{Definition} \footnote{Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.12 (Definition 1.2.3)}

For \( n = 1, 2, \ldots, N \), the price of the derivative security at time \( n \) if the outcomes of the first \( n \) tosses are \( \omega_1 \ldots \omega_n \) is defined to be the random variable \( V_n(\omega_1 \ldots \omega_n) \) of the above theorem. The price of the derivative security at time zero is defined to be \( V_0 \).

Price \( V_0 \) does not depend on coin tosses, and is the same for any path of stock price. In theory this is all you need to calculate the value of an option, but as was mentioned before, we need something to make the computational time shorter. Maybe the meaning of \( \Delta_n \) should also be explained. It is the number of shares that need to be held at time \( n \), to maintain the risk-neutral portfolio.

At this point we can discuss the establishment of a risk-free portfolio. The idea is in replicating the option with stock and money market investments, as we did in the simple and one period example, thus we hedge our position. In the multiperiod example this can be done with buying or selling shares of stock to establish a position with \( \Delta_n(\omega_1 \ldots \omega_n) \) shares of stock at time \( n \) and the rest of the money in the risk-free investment. This is called delta hedging and is a profitable business for banks.

\section*{4. Examples of option pricing in discrete models}

In order to successfully implement the idea of a multiperiod binomial model, we have to take advantage of its simplicity. Though a binomial tree with 100 generations ends in \( 2^{100} \) branches, we can dramatically cut the computational time if we set \( d = \frac{1}{u} \), giving us only 101 branch at the end of the tree. Thus we can simplify \( V_n(\omega_1 \omega_2 \ldots \omega_n) = V(S_n) \). This follows from an assumption that option price is only based on possible final outcomes, and not on the path from which they follow. This does not hold for the so-called \textit{lookback option}, or any other type of option, which is path dependant.

\subsection*{3.2 Simple example}

Let us just try out the method on a simple example, just to see how the model actually works.

Let us generate a tree of possible stock-price outcomes for the following input data. The starting price of the stock \( S_0 = \€ 37 \), movement per period \( u = 1.05 = \frac{1}{d} \), \( r = 0.12 \) (per 365 periods), the exercise price for the call option \( K = \€ 45 \) and the maturity of 100 periods. From this tree of possible prices we can calculate call option prices on each step, starting at time 100, and moving backwards to a single option price in present period. On the graph are the random path through the tree of stock prices and the prices of options at the selected vertices along the path. Even though the option prices...
clearly follow the movement of stock prices throughout the options’ lifetime, it is important to notice they do more so in the latter stage when any stock price change has less time to balance itself out. Also in the money peaks are less visible with out-of-the money ones because of the condition for payoff on the exercise day \( V_N = \text{Max}[S_N - K, 0] \). If we are deep out-of-the money close to exercise day, the option value will not react to stock moves and will be virtually zero (Picture 4). On the other hand, if we are deep in the money, option price reacts almost the same as the stock price (Picture 5). Of course the price of option for time 0 is not dependant on the path and is the same in every try.

![Picture 4: Out-of-the-money approach to expiration date (red and blue denote stock and call option prices respectively)](image)

![Picture 5: In-the-money approach to expiration date (red and blue denote stock and call option prices respectively)](image)

\(^9\) Picture 4 and 5 are simulations carried out by the author with an implementation of the binomial model
5. Value of an American call option

As a conclusion to this seminar let us just discuss an interesting question that arises when studying the basics of option valuation. As we have done all the calculations with the binomial model, assuming the option was European, we might ask ourselves how the calculation changes. It has been said before, that due to its extra feature the price of an American option should be greater or at least the same as the European one. One can be more flexible and that comes with a cost. Or does it? Let us examine this informally for both call and put options. For a reader that is more interested in mathematics behind this Appendix A contains a formal derivation of relation between European and American call option.

5.1 Informal evaluation of American option value

5.1.1 American call option

We will start with arbitrage bounds for call options. First let us note that

\[ 0 \leq V \leq S, \]

which means that the value of a call option cannot be negative nor can it be greater than the price of the asset (S) it is written for.

The important thing to note is that the next bound applies only to American call options. As they can be exercised at any given time, they are worth at least their intrinsic value, which is

\[ V = S - K. \]

The fourth bound is a little more subtle, and is the same for both types of call options. It has been illustrated in 3.2 in the apartment example. Let us say that we want to buy a share in one year’s time. Suppose the unknown share price is \( S^* \). We have two possibilities:

a. buy the share today for \( S \);

b. buy a one-year call on the share at a price of \( V \) and, at the same time deposit enough money at a risk-free rate to give us the exercise cost \( K \), in one year’s time. That will allow us to exercise the call if we choose so. The money we need to deposit is the present value of \( K \), \( PV(K) \).

Although both strategies guarantee the share in one year time, strategy (b) pays more in case that the share price finishes below the exercise price. In case that \( S^* \geq K \) both strategies pay \( S^* \), but if \( S^* \leq K \), strategy (b) pays \( K \). Because strategy (b) always pays as much and in some cases even more than strategy (a), it follows it must be worth more at time zero. Therefore:

\[ V + PV(K) \geq S, \]

and

\[ V \geq S - PV(K). \]

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Since we notice, when considering the third and fourth bound

$$[S - PV(K)] \geq (S - K),$$

we can see that the third bound is in effect redundant. This means that the call is always worth more than its intrinsic value, and the consequence is that there is no extra value in being able to exercise early. Since prices of both call options are the same, the same models can be applied for calculating them. A formal proof is given in the Appendix A.

### 5.1.2 American put option

Analogous to the call option example we come to the condition

$$0 \leq P \leq K,$$

where $P$ is the put price. In this case the upper bound is $K$ because that is what a put pays at maturity if $S$ was zero.

With a very similar example of put replication we come to the lower bound for both American and European put options

$$P \geq PV(K) - S.$$

But the American put carries an intrinsic value of $K - S$, and we can see that

$$PV(K) - S \leq K - S,$$

So that means that the lower bound for American put options is higher than for European. This is important as modifications to the model have to be made if we want to calculate the price of a call.

### 6. Conclusion

In conclusion we can look at the practical use of instruments presented in the seminar. Surprisingly, the simple binomial model with $u = \frac{1}{d}$ is quite commonly used probably for rough estimates, as the general version of the binomial model uses up to much computer time. One has do decide how much confidence he has in his model, as it will always provide an answer, but not necessarily the right one for his case.

The counter-weight to discrete simulation is the continuous-time Black-Scholes PDE and its derivatives. In theory, we get an instant answer, but the assumptions in the basic BS PDE are sometimes too rough and are much harder to implement in the continuous time model than in discrete. Some of the unconsidered events in the basic BS PDE are dividends, transaction costs, and the bid-ask spread, and are easier to incorporate in a simple binomial model.

Of course any model of option prices has to be tuned for every case, mostly because the volatility can be very different. Almost as much theory that there is for the models has been written for the choice of parameters.
5. Literature and references

5.1 Literature
2. Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005

5.2 References
A. Appendix (American and European call option value relation)

A.1 Basic theory

In order to derive the price of an American call option in relation to European one, we have to get familiar with a few definitions and theorems.

- **Definition of expectation**\(^{11}\)

Let \( X \) be a random variable defined on a finite probability space \((\Omega, \mathbb{P})\). The expectation (or expected value) is defined to be

\[
\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega) \mathbb{P}(\omega).
\]

When we compute the expectation using the risk neutral probability measure, we use notation

\[
\mathbb{E}^\mathbb{Q}X = \sum_{\omega \in \Omega} X(\omega) \mathbb{P}^\mathbb{Q}(\omega)
\]

(i.e. \( \Omega \) is a set of possible outcomes and \( \mathbb{P} \) is the probability measure).

Let us now remember the formula for stock price

\[
S_n(\omega_1 \omega_2 \ldots \omega_n) = \frac{1}{1+r} \left[ \bar{\beta} S_{n+1} (\omega_1 \omega_2 \ldots \omega_n H) + \bar{q} S_{n+1} (\omega_1 \omega_2 \ldots \omega_n T) \right].
\]

The term in brackets on the left side is obviously \( \mathbb{E}^n[S_{n+1}] (\omega_1 \omega_2 \ldots \omega_n) \). And we may write

\[
S_n = \frac{1}{1+r} \mathbb{E}^n[S_{n+1}].
\]

In this case there is an index \( n \) on \( \mathbb{E}^n \) which means that it is a conditional expectation, based on previous \( n \) coin tosses.

- **Conditional Jensen’s inequality**\(^{12}\)

If \( g(X) \) is a convex function of the dummy variable \( X \), then

\[
\mathbb{E}_n[g(X)] \geq g(\mathbb{E}_n[X]).
\]

- **Definition of martingales**\(^{13}\)

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\(^{11}\) Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.29 (Definition 2.2.4)

\(^{12}\) Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.34 (Theorem 2.3.2 (v))
Consider the binomial asset-pricing model. Let $M_0, M_1, ..., M_N$ be a sequence of random variables, with each $M_n$ depending only on the first $n$ coin tosses (and $M_0$ constant). Such a sequence of random variables is called an adapted stochastic process.

(i) If $M_n = \mathbb{E}_n[M_{n+1}]$, $n = 0, 1, ..., N - 1$, we say this process is a martingale.

(ii) If $M_n \leq \mathbb{E}_n[M_{n+1}]$, $n = 0, 1, ..., N - 1$, we say this process is a submartingale (even though it may have the tendency to increase).

(iii) If $M_n \geq \mathbb{E}_n[M_{n+1}]$, $n = 0, 1, ..., N - 1$, we say this process is a supermartingale (even though it may have the tendency to decrease).

Under the risk-neutral measure, the discounted stock price is a martingale.

**Definition of stopping time**\(^{14}\)

In an $N$-period binomial mode, a stopping time is a random variable $\tau$ that takes values $0, 1, ..., N$ or $\infty$ ($\infty$ meaning expiry without exercise) and satisfies condition that if $\tau(\omega_1 \omega_2 ... \omega_n \omega_{n+1} ... \omega_N) = n$, then $\tau(\omega_1 \omega_2 ... \omega_n \omega'_n \omega_{n+1} ... \omega'_N) = n$ for all $\omega'_n \omega_{n+1} ... \omega'_N$.

We define that notation $n \wedge \tau$ denotes the minimum of $n$ and $\tau$. We also define $S_n$ as a set of stopping times $\tau$ that take values from $\{n, n + 1, ..., N, \infty\}$. Hence $S_0$ is a set of all stopping times.

**Theorem-Optional sampling (Part II)**\(^{15}\)

Let $X_n$, $n = 0, 1, ..., N$ be a submartingale, and let $\tau$ be the stopping time. Then $\mathbb{E}X_{n\wedge\tau} \leq \mathbb{E}X_n$. If $X_n$ is a supermartingale, then $\mathbb{E}X_{n\wedge\tau} \geq \mathbb{E}X_n$ and if it is a martingale, then $\mathbb{E}X_{n\wedge\tau} = \mathbb{E}X_n$.

**European option price**\(^{16}\)

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\(^{13}\) Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.36

\(^{14}\) Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.97 (Definition 4.3.1)

\(^{15}\) Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.100 (Theorem 4.3.3)
Because European option price is a martingale, and the expected value of a martingale does not change through time we have

\[
\frac{V_n}{(1 + r)^n} = \mathbb{E}_n \left[ \frac{V_N}{(1 + r)^N} \right],
\]

which leads to another version of the pricing formula

\[
V_0^E = \mathbb{E}_n \left[ \frac{V_N}{(1 + r)^N} \right],
\]

for time zero. Note that \( V_N = \max [S_N - K, 0] \) for the call option.

- **American option price**\(^{17}\)

For each \( n, n=0,1,...,N \), let \( G_n \) be a random variable depending on the first \( n \) coin tosses. An American derivative security with intrinsic value process \( G_n \), if exercised at time \( n \) pays \( G_n \). We define the price process \( V_n \) for this contract by the American risk-neutral pricing formula

\[
V_n = \max_{\tau \in \mathcal{S}_0} \mathbb{E}_n \left[ \frac{1}{(1 + r)^{\tau - n}} G_\tau \right].
\]

The term \( \mathbb{I}_{\{\tau \leq N\}} \) means that for \( \tau = \infty \) the value is set to 0, as in this case there is no exercise.

**A.2 Derivation of American and European call option value relation**\(^{18}\)

Let \( g : [0, \infty) \rightarrow \mathbb{R} \) be a convex function and let \( g(0) = 0 \). This means that for \( s_1 \geq 0, s_2 \geq 0 \) and \( 0 \leq \lambda \leq 1 \) we have

\[
g(\lambda s_1 + (1 - \lambda) s_2) \leq \lambda g(s_1) + (1 - \lambda) g(s_2).
\]

Additionally we introduce a function

\[
g^+(s) = \max\{g(s), 0\},
\]

\(^{16}\) Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.40-41

\(^{17}\) Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.101 (Definition 4.4.1)

\(^{18}\) Shreve, Steven E., Stochastic calculus for finance I: The binomial asset pricing model, Springer science+business media, 2005, p.112
which, not by chance is familiar with the term for call payoff upon exercise. Function $g^+(s)$ also satisfies the same conditions as can be easily shown.

If we set $s_1 = s \geq 0, s_2 = 0$ and $0 \leq \lambda \leq 1$, we have

$$g^+(\lambda s) \leq \lambda g^+(s)$$

Because $\frac{1}{(1+r)^n} S_n$ is a martingale under the risk-neutral probabilities, we have $S_n = \mathbb{E}_n \left[ \frac{1}{1+r} S_{n+1} \right]$ and

$$g^+(S_n) = g^+ \left( \mathbb{E}_n \left[ \frac{1}{1+r} S_{n+1} \right] \right).$$

From the conditional Jensen’s inequality we obtain

$$g^+ \left( \mathbb{E}_n \left[ \frac{1}{1+r} S_{n+1} \right] \right) \leq \mathbb{E}_n \left[ g^+ \left( \frac{1}{1+r} S_{n+1} \right) \right].$$

And by taking $\lambda = \frac{1}{1+r}$ in (No123)

$$\mathbb{E}_n \left[ g^+ \left( \frac{1}{1+r} S_{n+1} \right) \right] \leq \mathbb{E}_n \left( \frac{1}{1+r} g^+[S_{n+1}] \right).$$

Putting the last three together, we see that

$$g^+(S_n) \leq \mathbb{E}_n \left( \frac{1}{1+r} g^+[S_{n+1}] \right),$$

and multiplication of both sides by $\frac{1}{(1+r)^n}$ yields the submartingale property

$$\frac{1}{(1+r)^n} g^+(S_n) \leq \mathbb{E}_n \left( \frac{1}{1+r} g^+[S_{n+1}] \right)$$

for the discounted intrinsic value process $\frac{1}{(1+r)^n} g^+(S_n)$. Because this process is a submartingale, (Theorem?) implies that for every stopping time $\tau \in \mathcal{S}_0$

$$\mathbb{E} \left( \frac{1}{(1+r)^N} g^+[S_N] \right) \leq \mathbb{E} \left( \frac{1}{(1+r)^N} g^+[S_\tau] \right) = V_0^F.$$

If $\tau \leq N$, then

$$\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1+r)^\tau} \ g(S_\tau) = \frac{1}{(1+r)^N} g(S_N) \leq \frac{1}{(1+r)^N} g^+(S_N)$$

and if $\tau = \infty$, then

$$\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1+r)^\tau} \ g(S_\tau) = 0 \leq \frac{1}{(1+r)^N} g^+(S_N)$$

In either case we have the same result, so we can write
\[ \mathbb{E} \left[ \mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1 + r)^\tau} g(S_\tau) \right] \leq \left[ \frac{1}{(1 + r)^{N\tau}} g^+(S_{N\wedge \tau}) \right] \leq V_0^E. \]

Since the last inequality holds for every \( \tau \in \mathcal{S}_0 \), we must have

\[ V_0^A = \max_{\tau \in \mathcal{S}_0} \mathbb{E} \left[ \mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1 + r)^\tau} g(S_\tau) \right] \leq V_0^E. \]

Since we have previously explained why American options must have at least the value of European options, we get that American and European call options are worth the same. One could say that the American call is worth more alive than dead as early exercise is in no case optimal.

This does not hold for American puts, because the discounted intrinsic value of a put is not submartingale. American puts are usually worth more than European and it might be optimal to exercise early.