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Faster than gravity

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Abstract

In the traditional ”faster than gravity” demonstration, a rigid stick is fixed to a horizontal table by a hinge at one end. A small ball is then placed on the stick. The stick and ball are now released at the same time and fall freely until they reach the table. Under certain conditions, the ball will reach the table after the stick, thus demonstrating that the stick fell faster than free fall.

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1. Introduction

Aristotle (384-322 BC) suggested that heavier object fall faster than lighter ones towards the Earth. Nearly 20 centuries later Galileo (1564-1642) experimentally disproved the long accepted claim by Aristotle that a falling object had a definite natural falling speed proportional to its weight. Galileo had found that the speed just keep on increasing, and weight was irrelevant. Galileo suggested that free from significant resistance, all objects fall with the same acceleration[1].
If we can neglect friction, all freely falling objects will fall at the acceleration of gravity, called *g*. If we drop a stick so that it rotates, at any instant one end will usually be accelerating at the rate greater than g while the other end will be accelerating at the rate less than *g*. A falling stick or board hinged at one end has similar characteristics. It is possible for the free end of the stick to accelerate at a rate greater than g. This effect has been demonstrated by the standard experiment called also The falling chimney. The acceleration of a rigid stick rotating around one end, and the time required for the stick to fall onto a horizontal table, is compared with the acceleration and falling time of a free particle originally placed on the stick. This experiment is also interesting because we can see comparison between two kinds of motion. This are translation of the ball and rotation of the stick.

2. The standard experiment

The apparatus used for this experiment is shown in Fig.1. Two sticks are fixed together by a hinge at one end. One stick is laid flat on the table and it is clamped along the edge of the table. The other stick is propped up by a stick so that it makes a small initial angle with the lower horizontal stick. If the prop is removed the stick will fall. It is possible for the free end of the stick to accelerate at a rate greater than *g*.

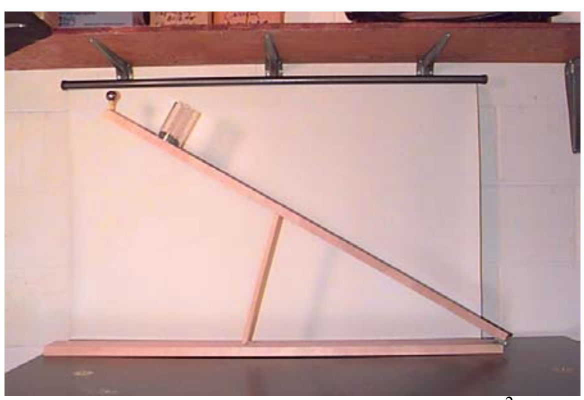

Fig.1: Demonstration of a ”faster than gravity” fall called falling chimney[2]

This effect can be seen by placing a steel ball bearing in a small indentation near the free end of the upper stick and a cup containing plasticine to catch the falling ball a short distance from the end. The position of the cup is dependent on the initial angle of the sticks but this can be fixed by using a prop of constant length. It is important that the lip of the cup be higher than the bottom of the ball initially so that it is clear that the cup has fallen under the ball. The presence of the ball in the cup after the demonstration is evidence that the vertical component of acceleration of the end of the stick was grater than the acceleration of gravity. Typical dimensions used in the experiment are stick length between 110,5 cm[3] and 121,9 cm[4] and stick width 3,8 cm[4] or 3,9 cm[3].

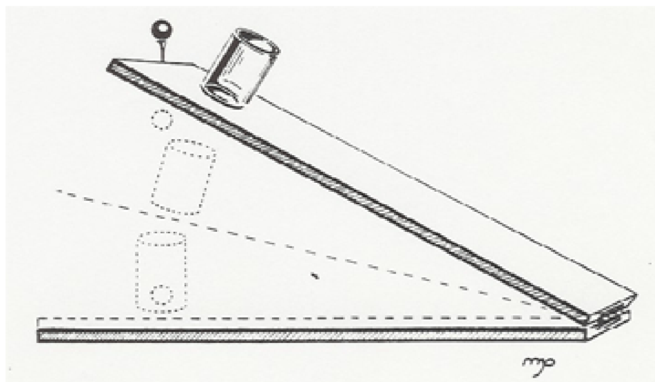

Fig.2: Fall of an inclined stick and a ball[5].

A qualitative explanation of the effect is that for a freely falling extended object such as a stick or board, its the centre of mass falls at *g*. Therefore if you drop two objects, their centres of mass will fall approximately at the same rate. In the experiment we assume, the centre of mass of the stick falls at the same rate as the ball. If the motion is constrained as in our demonstration, that is no longer true. But the centre of mass of mass still accelerates at almost *g*. Because the stick is stiff and hinged, the parts of the board farther from the hinge than centre of mass move faster. Since the centre of mass of the stick starts lower than ball, and they fall at the same rate, the stick will hit the table before the ball. So, even though the cup starts above the ball, it will hit the table first, and the ball can land in it (see Fig. 2). This explanation has also limits that we shall address in the next chapter. The visual quality and impact of the demonstration are greatly improved by use of a video camera and digital image storage for repetition in slow motion.
This experiment is called the ”falling chimney” experiment because when a large chimney is to be demolished a small explosive charge may be used to cause the chimney to tip over and fall to the ground. Photographs of such falling chimneys invariably show them breaking into two or more pieces before they hit the ground (see Fig. 3). This is a result of the centre of mass falling quicker than the upper end of the chimney.
The reason for this is that the brick structure does not have sufficient strength to force the top to have a vertical component of acceleration greater than g. As a result the top breaks away and falls with the acceleration g while the lower section is now a shorter uniform falling bar which is again subject to the same type of breaking.

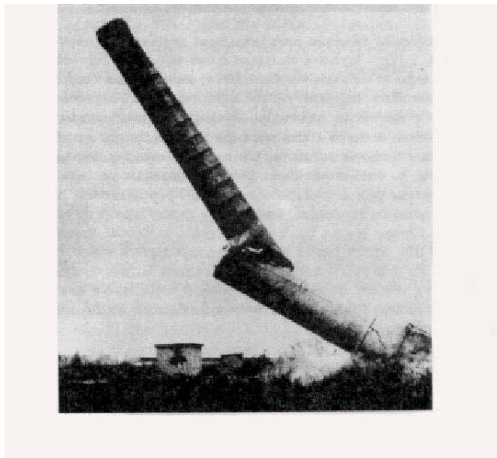

Fig. 3: Photography of the falling chimney[6].

3. The mathematical model and limits

This model assumes that there is no friction in the hinge, no air resistance and the stick is rigid. The geometry of the model is shown in Fig. 4. $\theta = \theta(t)$ is the angular displacement of the stick at time t, with θ_0 defining the initial position. The properties of the rigid stick are length L and mass M, with centre of mass a distance λL from the hinge and moment of inertia of the stick around a horizontal axis at the hinge given by $I = \beta ML^2$. The particle with mass m is placed at position P, a distance γL from the hinge. γ, β and λ are dimensionless constants. For the case of thin uniform stick $\lambda = ½$ and $\beta = ⅓$.

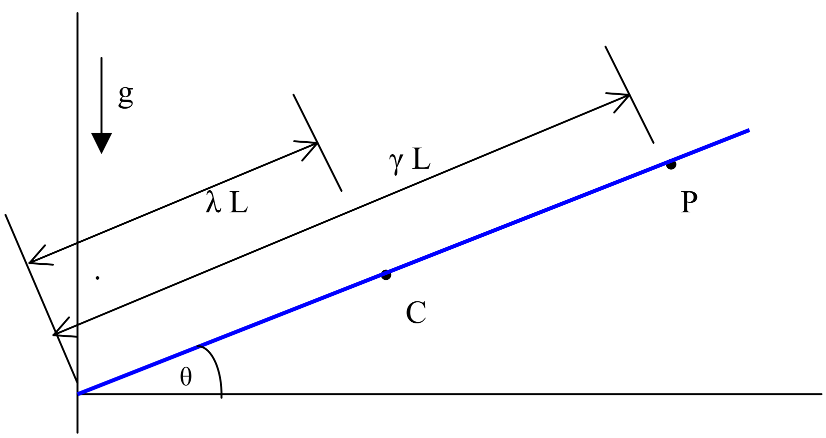

Fig. 4: Geometry of a falling stick of length L and mass M. Point C denote centre of mass, and point P denote position of the particle with mass m.

The model is analysed by applying Newton's laws of motion to the particle and the rigid stick separately[7,8]. All dissipative forces are ignored. The force of air resistance is less

than 3 % of gravity force. But if we want to find a size of all dissipative forces we have to compare the time a stick hit the table with the time found from this model.
The particle falls with a constant acceleration equal to g and reaches the table in time T_0, determined by

$$\gamma L \sin\theta_0 = \frac{1}{2} g T_0^2 \tag{1}$$

$$T_0 = \sqrt{\frac{2\gamma L}{g}\sin\theta_0} \tag{2}$$

Using second Newton's law for rotation of the rigid stick around a fixed point one obtains

$$\lambda LMg\cos\theta = \beta ML^2\ddot{\theta} \tag{3}$$

therefore,

$$\ddot{\theta} = \frac{\lambda g}{\beta L}\cos\theta \tag{4}$$

The angular velocity can be found from Eq. (4) by integrating once between θ_0 and θ, using the identity $\ddot{\theta}=\dot{\theta}\, d\dot{\theta}/dt$ and separating the variables, to give

$$\frac{1}{2}\dot{\theta}^2 = \int_{\theta_0}^{\theta} \frac{\lambda g}{\beta L}\cos\psi\, d\psi \tag{5}$$

or

$$\dot{\theta} = -\sqrt{\frac{2\lambda g}{\beta L}(\sin\theta_0 - \sin\theta)} \tag{6}$$

The time required for rotation of the stick from θ_0 to the horizontal position can be derived by integrating $\mathrm{d}t/\mathrm{d}\theta=\dot{\theta}^{-1}$ from Eq. (6)

$$T_1 = \sqrt{\frac{\beta L}{2\lambda g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin\theta_0 - \sin\theta}} \tag{7}$$

The tangential acceleration of the point P on the stick is:

$$a_t = \gamma L\ddot{\theta} = \frac{\gamma\lambda}{\beta} g\cos\theta \tag{8}$$

And its vertical component:

$$a_{ty} = \frac{\gamma\lambda}{\beta} g\cos^2\theta \tag{9}$$

The acceleration of the free end of the uniform stick ($\gamma = 1$, $\beta = ⅓$ and $\lambda = ½$) follows from Eq. (9):

$$a = \frac{3}{2} g\cos^2\theta \tag{10}$$

The special case $a = g$ leads to $cos^2\ \theta = ⅔$ and $\theta = 35^0$ 16’. This shows that for starting angles larger than 35^0 16’ ($cos^2\theta$ is monotonously descending function on the interval [0, π/2]) the free falling particle takes the lead at the beginning of the fall and may later be bypassed by the stick. Because of this reason in our case the initial inclination of the stick should be less than 35^0 16’.

In continuation we search the critical position of the particle. As a first criterion we can compare the particle’s acceleration with the initial acceleration of the point P, which is the point on the stick corresponding to the initial position of the particle. The ″faster than gravity″ case occurs when the stick moves away from the particle. This requires that $a_t > g\ cos\theta$ or from Eq. (8) $\gamma > \beta/\lambda$. The critical position is obtained as

$$\gamma^* = \frac{\beta}{\lambda} \tag{11}$$

In the case of the uniform stick the critical distance is a *2L/3* from the hinge. This criterion is not sufficient to ensure that the stick reaches the table before the particle does. Eq. 11 for critical position of the particle only applies when the stick is horizontal. The next criterion for ″faster then gravity″ case is to compare the times required to reach the table. That is if $T_1 < T_0$, where T_1 is the time for the stick to reach the table and T_0 (Eq. 2) is time for the particle. The integral for T_1 in Eq. 7 cannot be solved analytically, but can be transformed to an elliptic integral:

$$T_1 = \sqrt{\frac{\beta L}{\lambda g}} I_1(\theta_0) = \sqrt{\frac{\beta L}{\lambda g}} \int_{\Phi_1}^{\pi/2} \frac{d\Phi}{\sqrt{1 - k^2 \sin^2 \Phi}} \tag{12}$$

Where

$$\begin{aligned} \Phi &= \arcsin\left[\frac{1}{k}\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right] \\ k &= \cos\left(\frac{\pi}{4} - \frac{\theta_0}{2}\right) \\ \Phi_1 &= \arcsin\left(\frac{1}{\sqrt{2}k}\right) \end{aligned} \tag{13}$$

Now we found the critical position for this model:

$$\gamma^* = \frac{\beta}{\lambda} \frac{I_1^2}{2\sin\theta_0} \tag{14}$$

For all values of θ_0 Eq. 13 has to be used.

The time ratio T_0 / T_1 for this model is:

$$\frac{T_0}{T_1} = \sqrt{\frac{2\gamma\lambda}{\beta} \frac{\sin\theta_0}{I_1^2}} \tag{15}$$

For ″faster then gravity″ case this ratio has to be more than 1. Table 1 shows critical position and the time ratio for the free end of the uniform stick as a function of θ_0. For starting angles larger than 47,9º, the ball will hit the table first (see Table 1).

θ_0 (º)	γ^*	T_0 / T_1
0		$(3/2)^{1/2}$
1	0,6668	1,2246
10	0,6776	1,215
30	0,775	1,136
47,9	1,000	1,000
60	1,31	0,87

Table 1: Values of the critical position and the time ratio for the free end ($\gamma = 1$) of the uniform stick ($\beta/\lambda = 2/3$)[7].

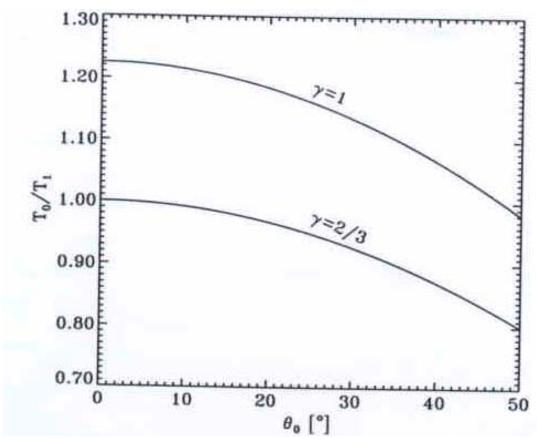

Fig. 5 : The ratio of transit times for a small ball to that of the stick as a function of the initial angle of the inclination of the stick with respect to the horizontal. The ball is initially placed at a distance γL from the hinge. Plots are shown for two values of γ: 1 (ball initially at the end of the stick) and 2/3 (ball initially at the centre of oscillation for small angles)[8].

The optimal angle of inclination for setting up the demonstration is not established from the ratio of transit times, but from their difference. For this purpose, we may define the dimensionless function for uniform stick

$$S(\theta_0,\gamma) \equiv \sqrt{\frac{g}{L}}(T_0 - T_1) = \sqrt{2\gamma \sin\theta_0} - \sqrt{\frac{2}{3}} I_1(\theta_0) \qquad (16)$$

The location of the peak is readily determined from the plot of S as a function of initial angle, for the appropriate value of γ (Fig. 6). The best display will be obtained for θ_0 = 22º15’, for which T_0 / T_1 is some 4 % below its peak value[8].

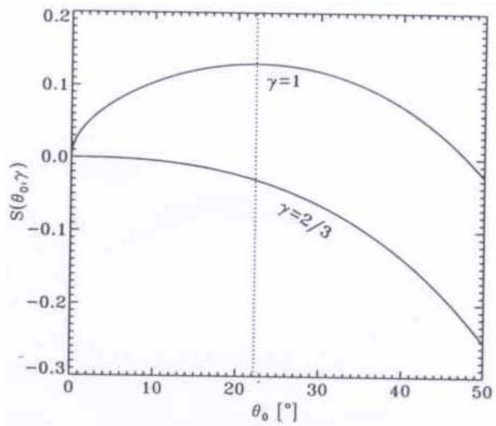

Fig. 6 : The difference function, Eq. 16, plotted as function of the initial angle of the inclination of the stick with respect to the horizontal. The ball is initially placed at a distance γL from the hinge. Plots are shown for two values of γ: 1 (ball initially at the end of the stick) and 2/3 (ball initially at the centre of oscillation for small angles). The plot for $\gamma = 1$ has a maximum at 22°15′. For $\gamma = 2/3$, the ball will always tend to fall faster[8].

The model we describe shows four cases.
Case 1: $\gamma < \beta/\lambda$ (2/3 for uniform stick). If the particle is closer the hinge than β/λ it remains in contact with the stick.
Case 2: $\gamma > \beta/\lambda$ (2/3 for uniform stick) and $\theta_0 < 35º$. The particle falls independently of the stick and reaches the table after the stick. In this case the stick is "falling faster than gravity".
Case 3: $\gamma > \beta/\lambda$ (2/3 for uniform stick) and $48º > \theta_0 > 35º$. The particle falls at the beginning faster than the stick, and may be later be bypassed by the stick. In this case the stick at the beginning decelerates the particle laid on it.
Case 4: $\gamma > \beta/\lambda$ (2/3 for uniform stick) and $48º < \theta_0$. The particle falls faster than the stick and hit the table first, or if at beginning laid on the stick it remains in contact with the stick.

4. The modified experiments

The apparatus used in modified experiment[8] has two indentations in the movable stick, at 16 mm and 400 mm from the free end of 1200 mm long stick, for carrying a pair of steel ball bearings. A small cup was positioned to catch the upper ball. An observer can compare and contrast the motion of objects in free fall with that of an extended body constrained at one point.
If the small weight is fastened to the free end of the stick[4] the tendency of this mass to have a free fall acceleration will reduce the acceleration of the end of the stick (Fig.7). If it is reduced enough the ball will not fall into the cup. In this case the ball remains in contact with the falling stick for a time after the release and will thus be given a velocity component away from the hinge during the first part of fall. This will cause the ball to miss the cup.

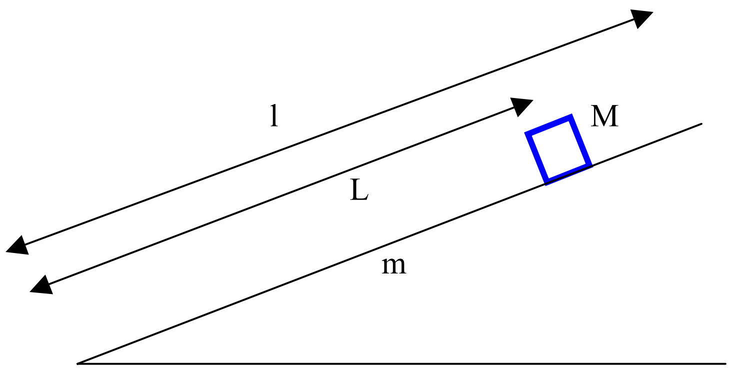

Fig. 7: The small weight fastened to the free end of the stick will reduce the acceleration of the end of the stick. *M* – mass of the weight, *m* - mass of the stick, *L* – distance from hinge to the weight, *l* – length of the stick

For a uniform stick of mass *m* with a weight mass *M* added to the end of the stick, the vertical component of acceleration of the end of the stick is:

$$a_{ty} = g\left[\frac{\frac{M}{m}+\frac{1}{2}}{\frac{M}{m}+\frac{1}{3}}\right]\cos^2\theta \qquad (17)$$

When $M \to 0$ we have the result given in Eq. (10) and as $M/m \to \infty$ we have $a = g\cos^2\theta$. Fig. 8 shows the relationship between θ and M/m of Eq. 17 for which $a = g$. For points above the curve $a < g$ and for points below $a > g$. For example, if $\theta_0 = 20^0$ and $M/m = 5$, the acceleration remains less than g until $\theta = 10^0$ after which acceleration becomes greater than g.

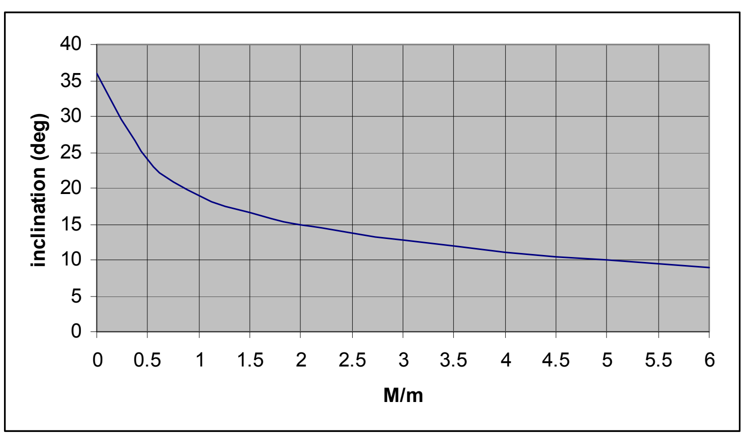

Fig. 8: Relation between inclination (θ) and *M/m* for which $a = g$.

The speed with which the board falls is dependent on how close the centre of mass of the board is to the hinge. The board will fall faster if the centre of mass is closer to the hinge.

One way of moving the centre of mass closer to the hinge is to change the shape of the board[9] (Fig.9).

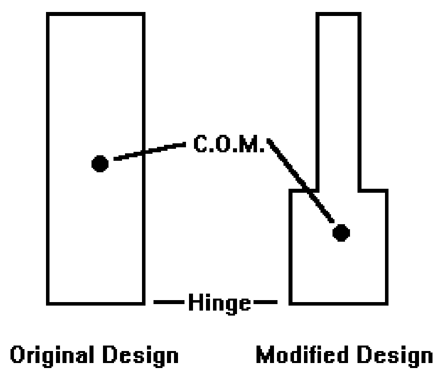

Fig. 9: The way of moving the centre of mass (com) closer to the hinge[8].

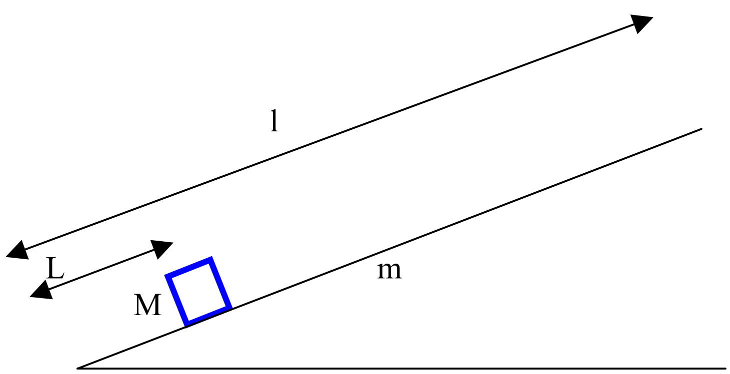

Fig. 10: The small weight fastened on the stick near the hinge will enhance the acceleration of the end of the stick. *M* – mass of the weight, *m* - mass of the stick, *L* – distance from hinge to the weight, *l* – length of the stick

Another way of moving the centre of mass closer to the hinge is to place a weight on the board near the hinge[10] (Fig. 10). The maximum acceleration of the free end of the board for combined system of the board and the brick on the board (their ends being coincident at the hinge) is:

$$a = \frac{3}{2} gl \left[\frac{ML + ml}{ML^2 + ml^2} \right] \qquad (18)$$

Where M and L are the mass and length of the brick, and m and l are the mass and length of the board. This is the acceleration when board obtains nearly horizontal position. Now we try to analyze the effects on changes in *M*, *L*, *m* and *l*. First we convert the equation in dimensionless form by introducing

$$\mu \equiv \frac{M}{m}, \lambda \equiv \frac{l}{L} \tag{19}$$

Then Eq. (17) becomes:

$$\frac{a}{g} = \frac{3}{2}\left[\frac{\lambda^2 + \mu\lambda}{\lambda^2 + \mu}\right] \tag{20}$$

The dependence of *a/g* on λ and μ in Eq. (20) is such that for any given value of μ, as λ varies, a curve results which passes through a maximum. This maximum occurs when $\lambda = 1 + (1+\mu)^{1/2}$.

First we consider an example of extreme conditions for μ and λ. Let the mass of board and the length of the brick approach zero while the length of the board and the mass of the brick approach infinity (very light and long board, very small and massive brick):

$$m \to 0, \quad L \to 0, \quad l \to \infty, \quad M \to \infty$$

If we now add the restriction that $ML = ml$, this implies that $\mu = \lambda$, and then, from Eq. (20):

$$a = 3g \tag{21}$$

The same result can be obtained by two other extreme choices of physical parameters for the system. The first choice consists of using the previous restriction that $ML = ml$, and letting $m \to 0$ and $l \to \infty$ as before but now keeping *M* and *L* both finite. The second choice consists of using the previous restriction that $ML = ml$, and letting $M \to \infty$ and $L \to 0$ but now keeping m and l both finite.

If $\lambda \gg \mu$ then, from Eq. (20):

$$a = \frac{3}{2}g \tag{22}$$

because the length of the board becomes the dominant factor.

Next we consider extreme conditions for μ, but not for λ. For example, let the mass of the board approach zero and the mass of the brick approach infinity while keeping the other parameters (both lengths) finite. Then we get from Eq. (20):

$$a = \frac{3l}{2L}g \tag{23}$$

We can also enlarged the dimensions and the movement documented by video because the result can easily be seen and repeated in slow motion. From a starting height of about 4 m the ball is about 1 m behind when the 6 m stick touches the floor[11].

5. Conclusion

In the experiment, the fall of two objects (an inclined rigid stick and a ball laying on it) is compared. The stick is falling faster than free particle under certain conditions. An investigation of the theoretical limitations assumes that there is no friction in the hinge and no air resistance. The falling stick impedes the free fall of the ball at the start for initial angles of inclination larger than approximately 35^0. The optimal initial angle of inclination of the stick is $22^0$15' for viewing the separate motion of a ball placed at the end. Even under optimal conditions, the visual quality and impact of the demonstration are greatly improved by use of a video camera and digital image store for repetition in slow motion. We can enhance or reduce this effect by adding mass on the stick.

The experiment described here is one of those we like to use to surprise our students. The result is demonstrated in a rather effective way. This demonstration also embraces a number of aspects of falling. It has the advantage that the level of explanation can be selected according to the audience.

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