Rainbow

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Abstract

In this seminar I will talk about rainbow. In the first part, I will discuss geometrical optical theory of the rainbow and with it try to explain some basic properties of this meteorological phenomena such as sequence of colours, the secondary bow, Alexander’s black band... In the second part of this seminar, I will take a look of the diffraction theory of the rainbow and prove the presence of supernumerary bows.

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1 Introduction

The rainbow is probably the best-known and the most interesting phenomenon of atmospheric optics and has excited attention from earliest times. The rainbow has an important position in physics. Some of the most powerful tools of mathematical physics were devised explicitly to deal with the
problem of the rainbow and with closely related problems. The rainbow has served as a touchstone for testing theories of optics. It is often thought that the scientific description of the rainbow is a simple problem in geometrical physics, which was solved long ago. This is not so. A satisfactory quantitative theory of the rainbow has been developed only in the last few years and involves much more than geometrical optics[1]. The first initial step in the theory of rainbow was taken by Descartes in 1637 in an appendix to his *Discours de la Methode*[2] where he said:" The rainbow is such a remarkable marvel of Nature... that I could hardly choose a more suitable example for the application of my method. "[1].

2 Observations

We see a rainbow in the sky usually only when the sun is behind us and we look at a cloud in front of us from which rain is falling, and is illuminated by the sunlight in certain positions. Rainbows are usually formed by raindrops rather than cloud droplets, though this is not universally the case. The bow itself appears as the arc of a circle, rarely complete, the centre of which is at the "anti-solar point", which represents the intersection of plane of the rainbow and the ray of light from the sun passing through the eye of the observer (See figure 1).

![Figure 1: First scheme shows the position of the sun and the observer so that he or she could see the rainbow. In the figure is also denoted the anti-solar point and position of supernumerary bows(Figure from[3]). The second picture shows the cover of Descarese Discours de la Methode.(Figure from[4])](image)

The rainbow contains all colours of the spectrum, more or less impure, with red lying on the outside of the arc and violet on the inside. The outer edge is more sharply defined than is the violet inner one and within the arc, the sky is brighter than it is without.

The principal bow just described is known as the primary bow. Outside it is often formed another bow, which is concentric to it. It is called the second bow and is usually much fainter than the primary. It consists of a similar range of colours as in the primary bow, but the order is reversed. Red appears on the inside and violet on the outside. The two rainbows are thus arranged red to red and between them, the sky is darker than it is either within the primary bow or outside the secondary. This dark band is often called the Alexander's dark band, from Alexander of Aphrodisias[5], who first remarked upon it.

Within the primary bow, there are often seen pale bows, usually faint pinks and greens but their colour is variable. These bands are called supernumerary bows(See figure 2.).
3 Geometrical Optics

3.1 Descartes theory

A basic analysis of the rainbow can be obtained by applying the laws of reflection and refraction[6] to the path of a ray through a droplet. Because the droplet is assumed to be spherical, all directions are equivalent and there is only one significant variable: the impact angle $\theta$ (See figure 3) or alternatively the impact parameter, which is defined as $b = \sin \theta$. At the surface of the droplet the incident ray is partially reflected and this reflected light we shall identify as the scattered rays of Class 1. The rest of light is transmitted with a change in direction caused by refraction. At the next surface light is partially transmitted (we get scattered rays of Class 2) and partially reflected. At the next boundary, the reflected ray is again split into reflected and transmitted components, and process continues indefinitely (See figure 3). The Class 1 rays represent direct reflection by the droplet and those of Class 2 are directly transmitted through it. Rays of Class 3 are those that escape the droplet after one internal reflection, and they make up the primary rainbow. The class 4 rays, which make two internal reflections, give rise to the secondary rainbow. Rainbows of higher order are formed by rays making more complicated passages, but they are not ordinarily visible.

Figure 3: Schemes show path of a monocromatic ray through a droplet.
Now we would like to calculate direction of class 3 rays in which they leave droplet. Therefore we define a quantity $D_1$ which tells us the deviation of outgoing ray (class 3) from direction of the incident ray. From figure 3 it is easy to see, that $D_1$ equals:

$$D_1 = \theta - \phi + \pi - 2\phi + \theta - \phi = 2\theta + \pi - 4\phi(\theta), \quad (1)$$

where we must remember, that angle $\phi$ is determined with the angle $\theta$ and Snell law. The deviation $D_1$ of scattered rays varies over wide range of values as a function of the impact parameter. As the impact parameter increases and incident rays are displaced from the center of the droplet, the scattering angle decreases. This trend, however, does not continue as the impact parameter is increased to its maximum value. Instead, the deviation $D$ passes through a minimum and thereafter starts to increase (See figure 4).

The deviation of higher order scattered rays (Class 4...) can be determined in a similar way as $D_1$ and can be written as:

$$D_i = 2(\theta - \phi) + i(\pi - 2\phi), \quad i = 2, 3, 4, ... \quad (2)$$

For rays of Class 4 the deviation $D_2$ is zero when the impact parameter is zero. As the impact parameter increases so does $D_2$ until it reaches maximum. After that, trend is reversed and scattering angle starts to decrease with increasing impact parameter (See figure 4).

![Figure 4: Graph shows how deviations of class 3 ($D_1$) and class 4 ($D_2$) rays change with the impact parameter $b$.](image)

In a sunlight the droplet is uniformly illuminated at all impact parameters at the same time. Therefore light is scattered virtually in all directions (See figure 7), but all scattered rays do not form a rainbow. If we want to see a rainbow, the intensity of the scattered light must be enhanced in one special direction (This special direction is called the rainbow angle and depends on the color of light.). Therefore rays with different impact parameters must be scattered at almost same angles. This can happen were the scattering angle varies most slowly with the impact angle $\theta$. That is of course around the extreme deviation $D_i(b)$, where $\frac{dD_i(b)}{db} = 0$. If we apply this condition to (1), we get the condition for the primary rainbow:

$$D_i^{extreme}(n) = 2\arcsin \left( \frac{\sqrt{4 - n^2}}{3} \right) + \pi - 4\arcsin \left( \frac{\sqrt{4 - n^2}}{3n^2} \right), \quad (3)$$

where is $n(\lambda)$ the refractive index of water for different colours of light. A condition for the second or higher rainbow can be derived in a similar manner from (2) and equals:

$$D_i^{extreme}(n) = 2\pi - \left[ i\pi + 2\arcsin \left( \frac{(i+1)^2 - n^2}{(i+1)^2 - 1} \right) - 2(1+i)\arcsin \left( \frac{1}{n} \sqrt{\frac{(i+1)^2 - n^2}{(i+1)^2 - 1}} \right) \right] \quad (4)$$
With (3) and (4) we are now able to calculate scattering angles for different colour bands of the primary and the secondary rainbow:

<table>
<thead>
<tr>
<th>Colour</th>
<th>$\lambda$ nm</th>
<th>$n_\lambda$</th>
<th>$D_{extreme}^1(n)$</th>
<th>$D_{extreme}^2(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>400</td>
<td>1.33141</td>
<td>139.57°</td>
<td>129.53°</td>
</tr>
<tr>
<td>Violet</td>
<td>700</td>
<td>1.24451</td>
<td>137.41°</td>
<td>126.14°</td>
</tr>
</tbody>
</table>

We can also calculate the width of both rainbows from difference between these angles:

$$\Delta D_1 = D_1(n_v) - D_1(n_r) = 1.88^\circ$$  \hspace{1cm} (5)

$$\Delta D_2 = D_2(n_v) - D_2(n_r) = 3.39^\circ$$  \hspace{1cm} (6)

From results we can see, that the second rainbow is almost twice as broad as the primary rainbow. We must be aware, that these results would be true only if rays of incident sunlight were exactly parallel. If we would want to be more precise, we would have to take into account the apparent diameter of the sun.

In figure 4 we can see that there are no rays of Class 3 and Class 4 scattered into the angular region between 130 and 138 degrees. This region is called Alexander’s dark band. If only rays of Class 3 and Class 4 would exist this region would be quite black. However rays of Class 1 and of higher order are scattered into this region. Therefore Alexander’s band is not totally black but still distinctly darker than the sky elsewhere (See figure 6).

### 3.2 Intensity of emerged light

Until now we have within geometrical optics shown how rays of light travel through a droplet and how because of different paths of rays with different wavelengths the rainbow is formed. We shall now take the theory of the rainbow a step further while still remaining within the same theory, and try to tell something about the intensity of scattered light.

The intensity of light which emerges from a raindrop after internal reflections depends at least of four factors: The coefficient of reflection and refraction at a water surface corresponding to various angles of incidence, the so called geometrical factor, the variation in the deviation in dependence of incidence angle ($\frac{dD}{d\theta}$) and diffraction and interference. Of all this four factors we have in theory of Descartes considered only the variation in the deviation. If we leave diffraction and interference for later discussion and consider only geometrical optics, we are left with first two factors to discuss.
3.2.1 Reflection and Refraction

The coefficient of reflection of light at a water surface depends upon the polarisation of the light. It is therefore necessary to treat light which is polarized in the plane of incidence (TE polarization) and that which is polarized at right angles (TM polarization), separately.

Using Fresnel Law[6], the intensity of reflected and transmitted light of TE polarization can be written as:

\[
R_\perp = \frac{\sin^2(\theta_r - \theta_i)}{\sin^2(\theta_r + \theta_i)} = \frac{I_r}{I_i} \\
T_\perp = 1 - \frac{\sin^2(\theta_r - \theta_i)}{\sin^2(\theta_r + \theta_i)} = \frac{I_t}{I_i}
\]

From graphs of (7) and (8) shown in figure 7, we can see, that for water with \( n = 4/3 \) at zero impact angle (\( \theta_i = 0^\circ \)) 98% of the light is transmitted and only 2% reflected. However for grazing incidence (\( \theta_i = 90^\circ \)) all light is reflected.

When the incident light is polarized at right angles to the plane of incidence, Fresnel found that
the intensity of the reflected and transmitted light was given by:

\[ R_{\parallel} \propto \frac{1}{I_{\parallel}^2} = \frac{\tan^2(\theta_i - \theta_r)}{\tan^2(\theta_i + \theta_r)} = \frac{I_r}{I_i} \]

(9)

\[ T_{\parallel} = 1 - \frac{\tan^2(\theta_i - \theta_r)}{\tan^2(\theta_i + \theta_r)} = \frac{I_t}{I_i} \]

(10)

Graphs of (9) and (10) are also shown in figure 7. At grazing incidence \((i = 90^\circ)\) again all the light is reflected and none refracted. At normal incidence \((i = 0^\circ)\) again 2 per cent is reflected, as with the light polarized in the plane of incidence. In the case of light polarized at right angles to the plane of incidence, however, a coefficient of reflection falls to zero when \(\theta_i + \theta_r = \frac{\pi}{2}\) since \(\tan(\theta_i + \theta_r)\) is then infinite. This phenomena is known as polarization by refraction[6] because, at that special Brewster angle\(^1\) only light polarized in the plane of incidence is reflected, that polarized at right angles is being totally refracted.

We may also notice, that the coefficients of reflection and refraction are unaltered if the angles of incidence and refraction are interchanged.

With these findings we are now able to calculate the intensity of light of the rainbow. Light which falls onto the primary rainbow is refracted on entry into the raindrop, is reflected at the back surface and is refracted again on emerging (See figure 3). If the angle of incidence of the light on the front surface is \(\theta_i\) and the angle of refraction is \(\theta_r\), then the angle of incidence under which the light strikes the back surface will be \(\theta_r\). Some light will be there refracted with the angle of refraction \(\theta_i\). The reflected light will then fall again on spherical surface under the angle of incidence \(\theta_r\). Light which will at that point come out of the droplet under refraction angle \(\theta_i\), will form the primary rainbow. Intensities of emitted TE and TM polarized light are therefore given by:

\[ I_{\perp} = \left(1 - \frac{\sin^2(\theta_r - \theta_i)}{\sin^2(\theta_r + \theta_i)}\right)^2 \frac{\sin^2(\theta_r - \theta_i)}{\sin^2(\theta_r + \theta_i)} \]

(11)

Reflection on back surface

\[ I_{\parallel} = \left(1 - \frac{\tan^2(\theta_r - \theta_i)}{\tan^2(\theta_r + \theta_i)}\right)^2 \frac{\tan^2(\theta_r - \theta_i)}{\tan^2(\theta_r + \theta_i)} \]

(12)

Reflection on back surface

The total intensity of light which forms primary rainbow is sum of both partial intensities:

\[ I^1 = I_{\perp} + I_{\parallel} \]

(13)

The intensities of TE and TM polarized light for secondary bow can be calculated in the same way. The only difference is that there are two internal reflections instead of one. This gives us:

\[ I_{\perp}^2 = \left(1 - \frac{\sin^2(\theta_r - \theta_i)}{\sin^2(\theta_r + \theta_i)}\right)^2 \frac{\sin^2(\theta_r - \theta_i)}{\sin^2(\theta_r + \theta_i)} \]

(14)

Two refractions

\[ I_{\parallel}^2 = \left(1 - \frac{\tan^2(\theta_r - \theta_i)}{\tan^2(\theta_r + \theta_i)}\right)^2 \frac{\tan^2(\theta_r - \theta_i)}{\tan^2(\theta_r + \theta_i)} \]

(15)

Two internal reflections

The total intensity is of course:

\[ I^2 = I_{\perp}^2 + I_{\parallel}^2 \]

(16)

\(^1\)For water, with a refractive index of 4/3, the Brewster angle \(\theta_B\) is of 53.13°.
How intensities of light which form the primary and the secondary rainbow depend of the angle of incidence $\theta_i$ is shown in figure 8. The angle of incidence corresponding to the primary rainbow is $59.3^\circ$ and is in figure marked by line labeled P. We can see, that the emergent light will consist of 96.2 per cent of light polarized in the plane of incidence and only 3.8 per cent of light polarized at right angles. This means, that the light of primary rainbow is strongly polarized.

The angle of incidence for the secondary rainbow is $71.83^\circ$. From corresponding graph we can see, that the light, which forms the secondary bow, consists of 90.3 per cent of TE polarized light and 9.7 per cent of TM polarized light. That means, that the secondary bow is less strongly polarized than the primary bow, but still quite polarized.

Eventually we can also calculate the ratio between intensities of both rainbows:

$$\frac{I(71.83^\circ)}{I(59.39^\circ)} = 0.0389 \div 0.09123 = 0.4263$$

This result tells us, that the secondary rainbow is much fainter than the primary rainbow.

### 3.2.2 Geometrical Factor

We shall treat this factor separately from reflection and refraction. These two coefficients will be added in the end as factors to final expression.

Let us consider a sunlight which falls upon the raindrop between angles of incidence $\theta_i$ and $\theta_i + \delta \theta_i$ (See figure 9). Between these angles of incidence the drop (with radius $a$) presents an area

$$\delta S = \pi a^2 \sin(2\theta_i) \delta \theta_i$$

(18)

to the incident light. Suppose that on emergence these rays lie between the angles $\theta_E$ and $\theta_E + \delta \theta_E$, measured from the direction from which the incident light comes (See figure 9). The solid angle into which the light is projected will be:

$$\delta \Omega = 2\pi \sin \theta_E \delta \theta_E$$

(19)

If $I_0$ is the intensity of light, which falls upon unit area at right angles to its direction of travel, then the intensity entering unit solid angle on emerging from the sphere after internal reflection will be:

$$I = \frac{I}{\delta \Omega} = I_0 \frac{a^2 \sin(2\theta_i)}{\sin \theta_E} \frac{\delta \theta_i}{\delta \theta_E}$$

Geometric fac. Cartesian fac.

(20)
Figure 9: First figure shows the cross-section which is illuminated by rays, which fall upon the drop between angles \( \theta_i \) and \( \theta_i + \delta \theta_i \). Second figure shows the solid angle into which this light is projected.

This intensity will determine the brightness of the image formed in the eye. When \( \frac{\delta \theta_E}{\delta \theta_i} = 0 \), we have the Cartesian condition for a rainbow.

Considering that angles of emergence of Class 3 and Class 4 rays are given by (See figure 10):

\[
\theta_{\text{Class 3}} = 4 \theta_r - 2 \theta_i = \theta_{E1} \tag{21}
\]
\[
\theta_{\text{Class 4}} = \pi + 2 \theta_i - 6 \theta_r = \theta_{E2}, \tag{22}
\]

then intensity of light, which is emitted from droplet after one or two internal-reflections, can be written as:

\[
I_1 = I_0 a^2 \frac{\sin(2 \theta_i)}{\sin \theta_{E1}} \frac{1}{\sqrt{n^2 - \sin^2 \theta_i} - 2} \tag{23}
\]
\[
I_2 = I_0 a^2 \frac{\sin(2 \theta_i)}{\sin \theta_{E2}} \frac{1}{2 - \frac{6 \cos \theta_i}{\sqrt{n^2 - \sin^2 \theta_i}}} \tag{24}
\]

Figure 10: Figures show paths of Class 3, rays which have one internal reflection, and of Class 4 rays, which have two internal reflections.

Now we have all the necessary parts (geometrical, reflection and refraction factor) to write final expression for the intensity of light which is emitted after reflections in the drop. If we multiply (23) by (13) we get the intensity of class 3 rays, which have one internal reflection:

\[
I_{\text{Class 3}} = I_0 a^2 \frac{\sin(2 \theta_i)}{\sin \theta_{E1}} \frac{1}{\sqrt{n^2 - \sin^2 \theta_i} - 2} \left( I_{\parallel} + I_{\perp} \right), \tag{25}
\]
where \( \theta_{E1} = 4 \arcsin \left(\frac{\sin \theta_i}{n}\right) - 2 \theta_i \). Similarly we can multiply (24) by (16) and get the intensity of class 4 rays with two internal reflections:

\[
I_{\text{Class4}} = I_0 a^2 \sin(2 \theta_i) \frac{\sin \theta_{E2}}{2 \frac{6 \cos \theta_i}{\sqrt{n^2 - \sin^2 \theta_i}} \left( I_2^\parallel + I_2^\perp \right)},
\]

(26)

where \( \theta_{E2} = \pi + 2 \theta_i - 6 \arcsin \left(\frac{\sin \theta_i}{n}\right) \).

From graph 11 we can see, how the intensity of Class 3 rays changes with \( \theta_{E1}(\theta_i) \). As the angle of incidence of the light increases from zero, the angle of emergence \( \theta_{E1} \), at first increases (That can be seen from figure 4 if we consider that \( \theta_{E1} = \pi - D(\theta_i) \)). At small \( \theta_{E1} \) the intensity of light is almost constant. When \( \theta_{E1} \) comes close to 40\(^o\), the intensity of light starts rapidly to increase. In first order theory it reaches a theoretically infinite value at \( \theta_{E1} = 42^o2' \) for light of refractive index \( n = \frac{4}{3} \). This is the angle at which we see primary rainbow. As the angle of incidence, \( \theta_i \), continuous to increase, both \( \theta_{E1} \) and the intensity of emerged light start to decrease again. However, by the time, the angle of incidence reaches the extreme value of 90\(^o\), \( \theta_{E1} \) has not again fall to zero. The smallest angle at which light emerges for large angles of incidence is \( \theta_{E1}^{\min} = 14^o \).

Similar curves can be drawn for Class 4 rays, which form the secondary rainbow. But in this case all the light is emerged at angles which are bigger than the angle at which we see the secondary rainbow. In figure 11 we can also see, that there is no emerged light between angles \( \theta_{E1} = 42^o2' \) and \( \theta_{E2} = 50^o58' \). This area is, as we already mentioned, known as Alexander's dark band.

![Figure 11: Graph shows how the intensity of emerged light of Class 3 (red line) and Class 4 (blue line) rays, which form the primary and the secondary rainbow, depend of angle of emergence \( \theta_E \). There is also clearly seen the Alexander's dark band, between angles \( \theta_{E1} = 42^o2' \) and \( \theta_{E2} = 50^o58' \).](image)

With knowledge that we have gained about Cartesian theory of the rainbow, we can simulate light scattering on the droplet and try to get a rainbow. Results of the simulation at different number of incident rays at different impact parameters are shown in figure 12.

### 3.3 Monte Carlo Computer simulation of the rainbow.

In the long history of scientific attempts to explain the rainbow, computer simulation is one of the most powerful modern methods[7]. Therefore it would be interesting, to make such a simulation, before going to the diffraction theory of the rainbow. This simple problem is perfect for illustrating the power of Monte Carlo techniques. The term "Monte Carlo" is derived from the famous casino in Monaco and is applied to any numerical method that makes frequent calls to a random-number generator.

The simulated rainbow is generated by plotting many single points in accordance with the laws of reflection and refraction applied to a large number of parallel rays of light incident on a sphere of water.
Figure 12: Pictures present 2D simulation of light scattering on a round droplet of water. Single ray scattering at $b = 0.75$ is shown in the first picture. Multiple ray scattering is shown in the second one.

Figure 13: Graph shows Monte Carlo computer simulation of the rainbow, produced by $N = 100000$ rays.
Results of the simulation are shown in figure 13. We can see, that computer from simple laws of geometric optics, acting on thousands of incidence rays, produces all features of actual rainbow. There are primary and secondary bow at correct angles with colours in the correct sequences, the diffuse white illumination just inside the primary, and the dark band between the primary and the secondary bow.

Once we have simulated a rainbow, made of water droplets, we can try to explore the effects on the dispersion of the colours, the sequence of colours, the angular sizes of the primary and secondary arcs, and the width of the dark band separating the primary and secondary arcs if droplets are not made of $H_2O$ but of some other substance. How the rainbow is changed if drops are made of concentrated (70%) $H_2SO_4$, or concentrated (70%) $HNO_3$ is shown in figure 14.

![Figure 14: Graphs show Monte Carlo computer simulation of the rainbow, produced by N = 100000 rays, where droplets are made of different substances.](image)

4 The diffraction theory of the rainbow

No rainbow is exactly like any other. Sometimes red or blue may be absent and the width of other color bands also variable. Sometimes we can not observe orange. In some bows the yellow band may be less broad then green and violet, whereas in others the yellow is much wider than green and violet. There is also a noticeable variation in intensities of different colours. Usually the violet side is the brightest but the colours there are more admixed with white. The total width of the rainbow is also not a constant.

Supernumerary bows, which are commonly seen within the rainbow, also differ from case to case. Sometimes many can be seen. Up to six have been observed. In some cases they are contiguous with the primary bow and in the other they are separated from it by a dark band. Their colours also vary. Green and pink are most common, but yellow and violet may also be seen.

All of these variations between rainbows are result of diffraction. The original diffraction theory of the rainbow was work of Sir George Airy in 1849 and Josef Pretner whose contribution to the subject was the detailed application of Airy’s calculation to the rainbow so as to disentangle the various colour formations predicted by the theory.

The diffraction effects arising in connection with the rainbow are connected with the light which emerges from a raindrop nearly in the direction of minimal deviation. As the point of entry, I, of the light into the raindrop (see figure 10) moves round in a clockwise direction from the point S, so as to increase the angle of incidence, the point of emergence, E, first moves in anti-clockwise direction, reaches an extreme position, and then moves back again. The point of interest to us in connection with the supernumerary bow is that the point at which E begins to retrace its steps lies beyond the point of minimum deviation $\theta_i^D$. For water of refractive index $n = \frac{4}{3}$, value of $\theta_i$ at which $\phi$ reaches its maximum value is $\theta_i^{\text{max}} = 76.84^\circ$. For minimum deviation in the rainbow the angle of incidence is, as we know, $\theta_i^D = 59.13^\circ$. That means, that point E passes the point of minimum deviation and
moves forward to the right in anti-clockwise direction until it reaches $\theta_i^{\text{max}}$.

Figure 15: Figure a) shows how rays of light at different impact parameters are being scattered in droplet. Dashed lines represent virtual rays. We get them with backward extrapolation of emerging rays. With help of virtual rays we calculate the wave front of emerged light, as shown in figure b).

Let us now replace the wave front emerging from droplet with a virtual source (See figure 15a). The position of the virtual source is arbitrary and has been chosen in the position before virtual rays intersect. We can now try to determine the function, that will satisfactorily approximate the real wave front.

In figure 15 b) DES represents the ray which emerges at minimum deviation. The rays both to the right ($\theta_i < \theta^P$) and to the left ($\theta^P < \theta_i < \theta^{\text{max}}$) of it, will be deviated through larger angles, as represented by lines ending in arrows. The emerged wave front has to be perpendicular to the rays and is represented by the curve WEW’. The curvature is concave upwards on the left and concave downwards on the right. Because the value of $y$ changes sign with $x$, the expression for $y$ will contain odd powers of $x$ only:

$$y(x) = fx + k'x^3 + ...$$

(27)

Since the curve WEW’ is at right angles to the axis of $y$ (consequently the slope of the curve WEW’ there is zero) when $x = 0$, the coefficient $f$ of the linear term must be zero. To a first order, therefore, the equation to the curve WEW’ must be:

$$y(x) = k'x^3.$$  

(28)

where $k'$ is constant. The linear dimensions of the curve will be proportional to the linear dimensions of the raindrop from which the wave front emerges, i.e. to the radius $a$ of the drop. The reciprocal value of the constant $k'$ must be of two dimensions in length, for dimensions of both sides of the equation to be the same. Therefore we can write:

$$y(x) = \frac{k}{a^2}x^3$$

(29)

where $k$ is now dimensionless constant. Using (29), the complex amplitude of the wave front may be than written as:

$$\Phi(x) = e^{-i\frac{2\pi}{\lambda}\sigma(x)},$$

(30)

where $\sigma(x)$ is the path difference between the disturbances from $M$, the coordinates of which are $x,y,$ and from $O$ in a direction making an angle $\theta$ with that of minimum deviation (see figure 16):

$$\sigma(x) = OS = OR - RS = OR - NT = x\sin \theta - y(x)\cos \theta = x\sin \theta - \frac{k}{a^2}x^3\cos \theta$$

(31)
Figure 16: Figure shows a path difference between the disturbances from M and O on given wave front.

With given wave front we are now able to calculate the disturbance formed at given ξ (See figure 16). If the observation point is far away from the droplet, we may use Fraunhofer diffraction formula[6] for our calculation. The disturbance at given ξ can be than written as:

$$\Phi(\xi) = e^{i \frac{2\pi}{\lambda L} \int_{-\infty}^{\infty} e^{-i \frac{2\pi}{\lambda L} \sigma(x)} e^{-i \frac{2\pi}{\lambda L} \xi x} dx},$$  

(32)

If we put (31) into (32), we get:

$$\Phi(\xi) = e^{i \frac{2\pi}{\lambda L} \int_{-\infty}^{\infty} e^{-i \frac{2\pi}{\lambda L} \sigma(x)} e^{-i \frac{2\pi}{\lambda L} \xi x} dx} \cdot \left(\frac{\lambda \alpha^2}{4k \cos \theta}\right)^{1/3} \int_{-\infty}^{\infty} \cos \left(\frac{\pi}{2} \left(\frac{u^3}{3} - zu\right)\right) du, \quad \text{(Rainbow Integral)}$$

(33)

where variable $z$ is defined as:

$$z^3 = \frac{16 \alpha^2 (\sin \theta + \tan \theta)^3}{\lambda^2 k \cos \theta} \quad \text{(34)}$$

Derived integral is known as Rainbow integral. When we calculate it, we get:

$$\Phi(\xi) = e^{i \frac{2\pi}{\lambda L} \int_{-\infty}^{\infty} e^{-i \frac{2\pi}{\lambda L} \sigma(x)} e^{-i \frac{2\pi}{\lambda L} \xi x} dx} \cdot \left(\frac{\lambda \alpha^2}{4k \cos \theta}\right)^{1/3} \cdot Ai \left(\frac{\pi^2}{12} \left(\frac{z}{\lambda \alpha^2 / k \cos \theta}\right)^{1/3}\right),$$  

(35)

where $Ai(x)$ is famous Airy function[8]. A graph of the rainbow integral is drawn in figure 17. It resembles the curve for the diffraction pattern near the edge of the shadow of a straight edge in a very striking way. The illumination at the position of minimum deviation at $x = 0$, corresponding to Descartes ray of the simple rainbow theory is 0.44 of the first maximum. From graph we can see that, as with the shadow of the straight edge, there is some illumination within the area from which light should be cut off according to geometrical optics. Within this area illumination falls rapidly when we go further away from the geometrical edge of illumination. Again, as with the diffraction round a straight edge, in the area which is illuminated according to geometrical optics, there occurs a series of fringes. As observed in the sky they lie within the ark of the primary rainbow, and the bright fringes give rise to the supernumerary bows. The first bright fringe is the most intense and is origin of the primary bow. This fringe is than followed by a long series of others of only slowly
Figure 17: Graphs show the square of the rainbow integral. It resembles the curve for the diffraction pattern near the edge of the shadow of the straight edge.

diminishing intensity. As one proceeds into the illuminated area the minima become closer together and the resulting coloured arcs will become narrower. In contrast to the pattern produced by a straight edge, there is no illumination in the dark fringes. This corresponds to the fact that at large distances from the rainbow the intensity is low.

As mentioned, the derived equation for the emergent wave front is only a first approximation and will be valid only for points near the ray with minimal deviation. While applying our theory, we have made an assumption that diffraction arises from points on the wave front near to the Descartes ray. Therefore Airy’s theory holds only if the drops are large compared to the wavelength of light. Rain drops which form rainbow normally satisfy this condition. However, even in the case of smallest cloud droplets their diameter is of order of ten wavelengths so that even here a useful quantitative picture should still be obtainable.

The rainbow integral, which we have been discussing so far, gives the intensities to be expected in various directions, as long as the drop size remains the same and the wavelength of the light is always the same. The constant factor:

$$\left( \frac{4\pi^2}{3} \frac{a^2}{k\lambda^2 \cos \theta} \right)^{1/3},$$

by which the Airy function in (35) is multiplied, suggests that as the wavelength and the radius of the drop are altered, the amplitude will vary as $$\left( \frac{a}{\lambda} \right)^{2/3}$$, considering that $$\theta$$ is always small and its cosine approximately equal to unity.

The described calculation of Airy’s theory has been so far made in two dimensions. However, the real problem is, of course, one in three dimensions. If the radius of the drop from which the light is emerged is large compared with the distance along the wave front over which the integration has to be preformed in order to calculate the intensity of the diffracted light, the integral itself will not be affected. But it is necessary to multiply the constant factor (36) by a further $$\sqrt{\frac{\pi}{\lambda}}$$ (See[2]). The intensity, which varies as $$\Phi^2$$ will, therefore, be proportional to:

$$I \propto \left( \frac{a}{\lambda} \right)^{7/3}$$

Raindrops have generally diameter greater than 0.5mm and range in size up to few millimeters.
This result will be required, when we will compare intensities of various colours (i.e. with different $\lambda$) and when we will take a look of the effect of variations in the sizes of the drops.

### 4.1 Colours in rainbow

In our diffraction theory of rainbow we have been dealing so far only with a monochromatic light. We know that in the rainbow all spectral colours are present and superimposed upon each other. Each colour will possess its own characteristic angle of minimum of deviation, its own characteristic intensity in the spectrum and consequently its own diffraction curve, in its own position, and with its own characteristic amplitude. The problem in providing a theoretical analysis for the rainbow, is thus to work out the effect of such superimposition.

In my own analysis I have followed the investigation of J.M. Pernter, who first solved this problem and made an important contribution to the optics of the rainbow. I have started my analysis with calculation of the angle of minimum deviation for each colour (wavelength). With that information I was than able to determine the positions in which the various Airy’s curves have to be arranged.

![Airy Curves](image)

Figure 18: First figure shows Planck’s function for a blackbody radiation. Second one shows various Airy curves for different wavelengths, when the sun radiates as a blackbody.

I have also taken into account that the Amplitude of each Airy’s curve must be proportional to the square root of the intensity of the light of the corresponding colour in the sunlight. I have presumed that the sun radiates as a blackbody with $T = 5780K$ and, therefore, obeys Planck radiation[9] formula:

$$\frac{dI}{d\lambda} = \frac{2\pi hc^2}{\lambda^5} \left( \frac{1}{e^{\frac{hc}{\lambda kBT}} - 1} \right)$$

(38)

Results I have got are shown in figure 18. We can see, that in case of the sun being a blackbody the violet light has the greatest intensity in the rainbow. From observations we know, that this is not true. To get more accurate results, we have to consider the scattering of sunlight off particles of the atmosphere. The intensity of the transmitted light is given by:

$$I(\lambda) = I_0(\lambda)e^{-\frac{A}{\lambda}}$$

(39)

where $I_0(\lambda)$ is the intensity of the sunlight before scattering in the atmosphere. The parameter $A$ depends on the mean thickness of the atmosphere, the speed of light and of the total cross-section of the scattering of light off the particles in the air. From (39) we can see, that strong wavelength dependence of scattering enhances the long wavelengths and reduces the short wavelengths. When we multiply the amplitude of each Airy’s curve with this additional factor, we get results shown in figure 19.

Once I had all individual Airy curves for every color, I superimposed them upon each other and got the final theoretical picture of the rainbow. Results of my analysis for different sizes of droplets...
Figure 19: Figure shows various Airy curves for different wavelengths, when scattering of light off the particles in the air is considered. I have set parameter $A$ to $A = 4 \times 10^{10} \text{nm}^4$

are shown in figure 20. From pictures we can see how properties of the rainbow (width of the bow, colours, number of visible supernumerary bows...) depend of the radius of droplet. In simple Descartes theory the rainbow was independent of the radius of raindrops, but now we can see (from Airy's theory) that the size of the drops matters and is an important parameter of the rainbow.

5 Raindrop size distribution

In the end it would be right to say a few words about the distribution of raindrop size. It is described by empirical Marshall-Palmer distribution, which is a negative exponential function:

$$N(D) = N_0 e^{-\Lambda(R)D}, \quad (40)$$

$N(D)$ tells us the number of drops with diameter $D$ and $N_0$ is the number of intercepted raindrops. The empirical parameter $\Lambda(R)$ determines the slope of exponential curve and is given by:

$$\Lambda(R) = 41 R^{-0.21}, \quad (41)$$

where $R$ is rainfall rate in millimeters per hour. Graphs of Marshall-Palmer distribution for different rainfall rates are shown in figure 21.

6 Conclusion

The theory of the rainbow described in this seminar explains formation of the rainbow in terms of classical rays of light and Airy theory which is based on the interference between two rays of light traveling in the same direction. In terms of these theories we could also explain similar phenomenon like white bow, rainbow by reflected light, moonbow and halo[2]. However, these theories are not the whole story of light scattering by water droplets, and they cannot begin to explain another phenomenon involving light scattering from mist: the glory. To understand glory, the full scattering theory of light by a dielectric sphere, Mie theory, is needed. In spite of that, the beauty of the physics of the rainbow is that so much can be understood in terms of rays and simple wave theory. Rainbows open our eyes to some of the fundamental properties of light.

7 Thanks

I would like to thank to my mentor doc. dr. Igor Poberaj for all support and help he has given me, while I have been preparing this seminar.
Figure 20: Figures show theoretical rainbows calculated from Airy’s rainbow theory for different raindrop sizes.
Figure 21: Figure shows Marshall-Palmer distribution of raindrop size for different values of the parameter $R$ (rainfall rate).

References


